Notes to GEOF346
Tidal Dynamics and Sea Level Variations

Fall 2022

Helge Drange
Geophysical Institute, University of Bergen
helge.drange@gfi.uib.no

Mainly off-hour typing – so give a word regarding errors, inconsistencies, etc...
Also give a word in case of specific requests (derivations, unclear issues),
and I will try to update the notes accordingly.

The note will be regularly updated – so keep printing at a minimum...

The notation (closely) follows that used in the text book:
Sea-Level Science by D. Pugh and P. Woodworth, Cambridge, 2014
(`https://doi.org/10.1017/CBO9781139235778`)

Parts of the derivations follow
Forelesninger over tidevann og tidevannsstrømmer. Teori, opptreden, analyse og prediksjon,
by Herman G. Gade, Geofysisk institutt, Universitetet i Bergen, 1995

Note:

- Symbols/expressions in red are, to the best of the author's knowledge, corrections to typos in the text book.
- For simplicity, the notation

$$\sin\, 2\,a \overset{(628)}{=} 2\sin a\cos a$$

  implies that, in this case, equation (628) has been adopted in the displayed expression.
- Norwegian word/phrases are given in quotation marks in footnotes.
- A (far from) complete Norwegian dictionary is provided in appendix I.
- ...and the index section is only slowly being updated.
- Last, but not least: Comments to the compendium are always appreciated!

October 25, 2022

Contents

[1]Mainly based on R. Fitzpatrick (): *An Introduction to Celestial Mechanics*, M. Hendershott (): *Lecture 1: Introduction to ocean tides*, and Murray & Dermott (1999), *Solar Syatem Dynamics*, Chap. 2.

[2]Mainly based on the excellent treatise by M. Capderou (2014): *Handbook of Satellite Orbits. From Kepler to GPS*, ISBN 978-3-319-03415-7, DOI 10.1007/978-3-319-03416-4, Springer International Publishing Switzerland.

Part I

Tides, basic astronomical periods and frequencies

1 Supporting literature

1.1 General text

- M. Hendershott, Introduction to ocean tides, 2004
`https://www.whoi.edu/cms/files/lecture01_21351.pdf`
(from `https://gfd.whoi.edu/gfd-publications/gfd-proceedings-volumes/2004-2/`).
- B. Simon, *Coastal Tides*, 2013
`https://iho.int/iho_pubs/CB/C-33/C-33_maree_simon_en.pdf`.
- Z. Kowalik and J. L. Luick, 2019, *Modern Theory and Practice of Tide Analysis and Tidal Power*
`https://uaf.edu/cfos/files/research-projects/people/kowalik/Book2019_tides.pdf`.

1.2 Tidal energy

Z. Kowalik, *Tide distribution and tapping into tidal energy*, 2004
`https://uaf.edu/cfos/files/research-projects/people/kowalik/tides04.pdf`.

2 Definition of the tides

Tides can be defined as (Gregory *et al.* (2019), *Surv. Geophys.*, **40**, 1251–1289):

> *Periodic motions within the ocean, atmosphere and solid Earth due to the rotation of the Earth and its motion relative to the moon and sun. Ocean tides cause the sea surface to rise and fall.*

From the above, the astronomical configuration is of key importance to understand the tides. The following illustrations, with characteristic periods, provide a schematic overview of the Earth-Moon-Sun system (will be updated...). For further explanations and detailed derivations, see e.g. Chap. 3 in the textbook or Sec. 8 below.

3 Solar periods and frequencies

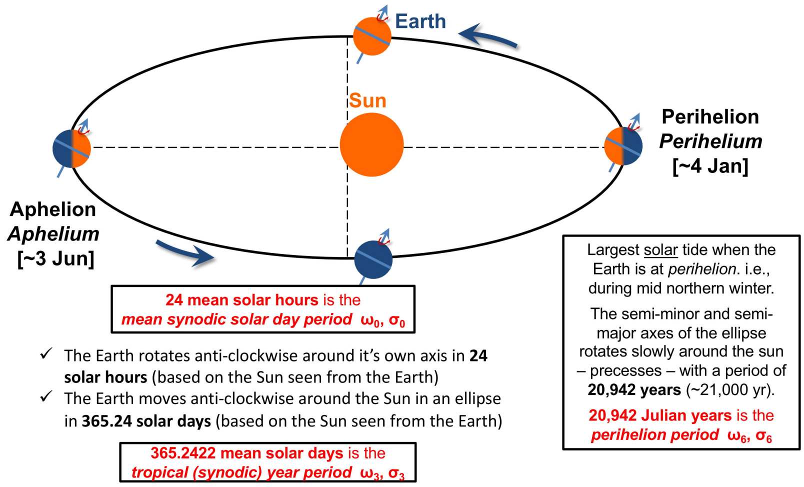

Figure 1: Schematic overview of the Earth-Sun system.

4 Lunar periods and frequencies

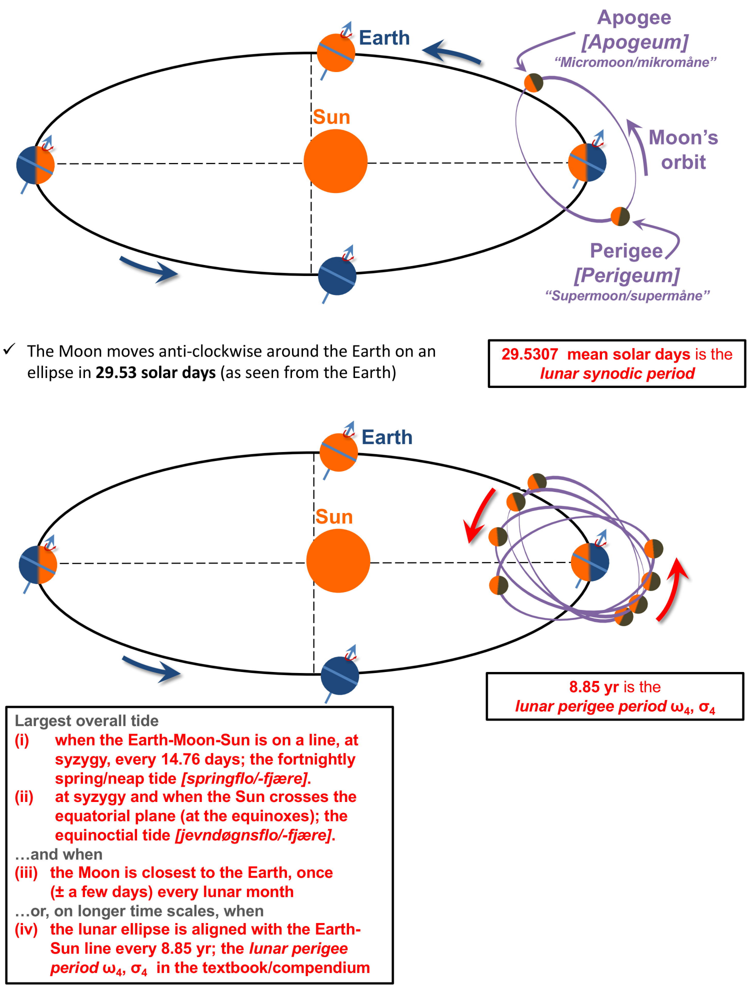

Figure 2: As Fig. 3, but including the Moon. In addition to the lunar synodic period, the lunar sidereal month with a period of 27.3217 mean solar days are denoted ω_2 , σ_2 . The period of the spring/neap tide – of 14.76 days – are derived in Sec. 10.1.

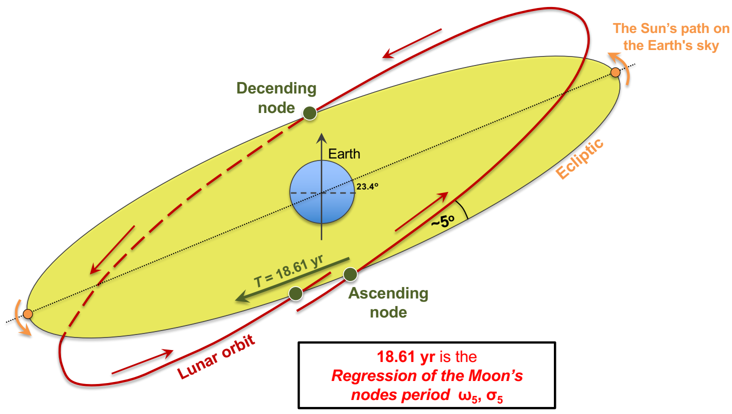

Figure 3: Schematic overview of the regression of the Moon's nodes with a period of 18.61 yr.

Part II

Equilibrium theory and derivations thereof

5 Equilibrium theory, direct method

A key objective in ocean tidal dynamics and analysis is to derive an analytical form of the gravitational attraction on the global ocean caused by the presence of the Moon and the Sun. Once this force is found, it can be added as a forcing term to the momentum equations which, together with the continuity equation, model the full 2- and 3-dimensional flow of the global ocean tide.

The derivation of the resulting *Equilibrium Tide* and subsequent analysis are addressed in Part II. A description of the basic tidal wave characteristics is also provided (Part IV). Modelling of the full 2- and 3-dimensional dynamics of the ocean tide is beyond the scope of the presented lecture notes (but might be added at a later stage).

In the following sections, the gravitational force is derived based on what one may phrase the direct method. An alternative approach – commonly used in tidal dynamics – derives the gravitational force from the tidal potential. The derivation of the latter is given in Sec. 13.

5.1 Geometry of the Earth-Moon system

The configuration of the Earth-Moon system used for deriving the properties of the *tidal equilibrium* is displayed in Fig. 4. It follows from the figure that

$$\mathbf{r} + \mathbf{q} = \mathbf{R} \tag{1}$$

or, to explicitly label the Earth-Moon system, with the subscript l for lunar,

$$\mathbf{r} + \mathbf{q}_l = \mathbf{R}_l \tag{2}$$

5.2 Centre of mass of the Earth-Moon system

The centre of mass of the Earth-Moon system is located along the centre line OM at a distance $x\,R$ $(0 < x < 1)$ from point O in Fig. 4. We then get that

$$m_e\, x\, R_l = m_l\, (1 - x)\, R_l \tag{3}$$

or

$$x = \frac{m_l}{m_e + m_l} \approx 0.012 \tag{4}$$

Here m_e and m_l are the mass of the Earth and Moon, respectively, see appendix A for numerical values. With mean values of r and R (see appendix A), we get that

$$x \approx 0.73\, r \tag{5}$$

implying that the centre of mass of the Earth-Moon system is located about three quarters of Earth's radius from the centre of the Earth.

The centre of mass of any two- or multiple-body system is also called the system's *barycentre*.

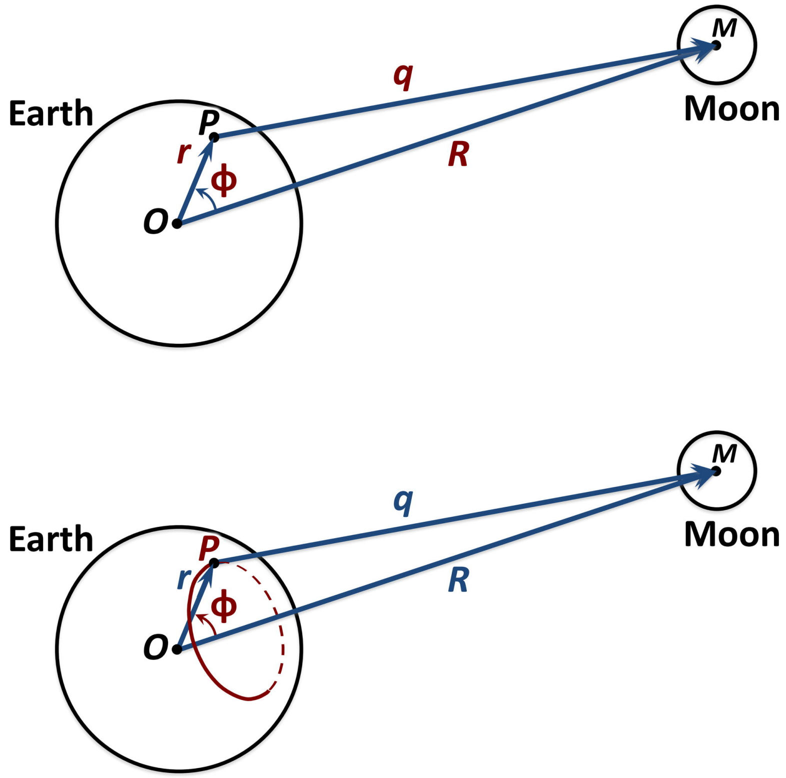

Figure 4: Illustration of the Earth-Moon system with the Earth to the left and the Moon to the right (figure greatly out of scale). O, P and M are the centre of the Earth, an arbitrary point on Earth's surface and the centre of the Moon, respectively. *Upper panel:* $\mathbf{r}$ is the Earth's radius vector (from point O to P), $\mathbf{R}$ is the position vector from the centre of the Earth to Moon's centre (from O to M), and $\mathbf{q}$ is the position vector from P (on Earth's surface) to M. The line between O and M is sometimes called the *centre line* and the angle ϕ the *zenith angle* or the *centre angle*. A similar configuration holds for the Earth-Sun system. *Lower panel:* As above, but illustrating the circle on the Earth's surface spanned out by P for $\phi = \text{const}$.

5.3 Newton's law of universal gravitation

Newton's law of universal gravitation (from 1687) states that an attractive force F is set up between any two point masses, varying proportionally with the product of the masses (m_1 and m_2) and inversely proportional with the distance R between the masses **/3.2/, note R!**

$$\boxed{F = G\,\frac{m_1\,m_2}{R^2}} \tag{6}$$

Here $G = 6.674 \times 10^{-11}$ N m^2 kg^{-2} is the gravitational constant.

5.4 Gravitational forces and accelerations in the Earth-Moon system

The gravitational force at the Earth's centre because of the presence of the Moon, $\mathbf{F}_l$, is

$$\mathbf{F}_l = G\,\frac{m_e\,m_l}{R_l^2}\,\frac{\mathbf{R}_l}{R_l} \tag{7}$$

where $\mathbf{R}_l/R_l$ is the unit vector along the Earth-Moon centre line.

According to Newton's second law, $\mathbf{F} = m\,\mathbf{a}$, this force leads to an acceleration of the centre of the Earth $\mathbf{a}_O$ according to

$$\mathbf{a}_O = \frac{\mathbf{F}_l}{m_e} = G\,\frac{m_l}{R_l^2}\,\frac{\mathbf{R}_l}{R_l} \tag{8}$$

Similarly, the gravitational acceleration on a body with mass m_P at point P caused by the Moon is

$$\mathbf{a}_P = G\,\frac{m_l}{q_l^2}\,\frac{\mathbf{q}_l}{q_l} \tag{9}$$

At point P, there is also a gravitational acceleration g towards the centre of the Earth caused by Earth's mass, on the form (9):

$$\mathbf{g} = -G\,\frac{m_e}{r^2}\,\frac{\mathbf{r}}{r} \tag{10}$$

By inserting the numerical values of G, m_e and r (see appendix A) in (10), one obtains

$$g = 9.8 \text{ m s}^{-2} \tag{11}$$

as expected. Furthermore, the absolute value of (10) leads to the relationship **/3.4/**

$$\boxed{G = g\,\frac{r^2}{m_e}} \tag{12}$$

5.5 Earth's movement around the Earth-Moon centre of mass

To maintain the Earth-Moon centre of mass at $0.73\,r$, the Earth can either *rotate* around the joint centre of mass as a solid body, or the Earth can adjust it's position around the common centre of mass without rotation, named *revolution without rotation*.

The former would imply that the centrifugal acceleration imposed on the Earth varies constantly and everywhere on the Earth (surface as well as the interior), involving constantly converging and diverging strain forces. This is not the case. In stead, *revolution without rotation* takes place as outlined in Fig. 5; yielding identical centrifugal forces at any point on the surface and in the interior of the Earth.

An alternative way to illustrate the revolution motion around the common centre of mass of a two-body system is given by an animation at

`http://folk.uib.no/ngfhd/Teaching/Div/revolution_circle.gif`.

In this case, the two-body system consists of a major (grey coloured) and a minor (blue) body encircling the former (and ignoring, for the time being, that the masses may rotate around their own rotation axis). The red point is fixed to the surface of the main body and it points towards left at all times, illustrating the *revolving* movement around the common centre of mass.

The above implies that every point on solid Earth describes a circular motion with radius

$$s = 0.73\, r \tag{13}$$

but with different centres, and with the rotation rate ω governed by the rotation rate of the Moon around the Earth. Consequently, each point on Earth, whether on Earth's surface or in the Earth's interior, will experience identical centrifugal acceleration (i.e., an outward-directed acceleration relative to the rotation of the Earth-Moon system) with magnitude (M & P (2008), eqn. 6.28)

$$a_\omega = \omega^2\, s \tag{14}$$

5.6 A (somewhat complicating) side note

The actual movement of the Earth-Moon system is a little more complicated than that described above. A more correct picture, albeit heavily out-of-scale for the sake of illustration, is shown in the following animation:

`http://folk.uib.no/ngfhd/Teaching/Div/revolution_ellipse.gif`.

According to *Kepler's laws for a two-body system* (see Section 15), the Moon (blue disc) moves around the Earth in an *ellipse* (blue curve), with the Earth located at one of the ellipse's focus points (right-most, large cross); the Moon moves fast when it is closest to the Earth (perihelion) and slow when it is far away from the Earth (aphelion); and the Earth adjust it's position relative to the Moon by *revolution without rotation*, as described in the introduction to the previous section.

The Earth-Moon configuration also holds for the Earth-Sun system, now with the Earth moving around the Sun along an ellipse, with the Sun located in one of the ellipse's focus points. Since the Earth's speed around the Sun varies with the distance to the Sun, there is a mismatch between the time shown on a sundial on the Earth (which is determined by the Earth's exact position relative to the Sun) and the time on any mechanical or electronic clock (which is based on the assumption that the Earth encircles the Sun with constant speed). This mismatch amounts to ± 15 min during the course of a year, and is described by the so-called *Equation of time* (Section 15.8).

Accurate tidal analysis or computation needs to take into account the above factors; that the path of the orbiting body follows an ellipse with a constantly varying distance between the two bodies, and that both the speed and time vary during the orbiting period. At lowest order, the Earth-Moon and the Earth-Sun distance, as well as the time on the Earth relative to the Sun, can be considered constant. The basic influence of the Moon and the Sun on the Earth's tidal variations

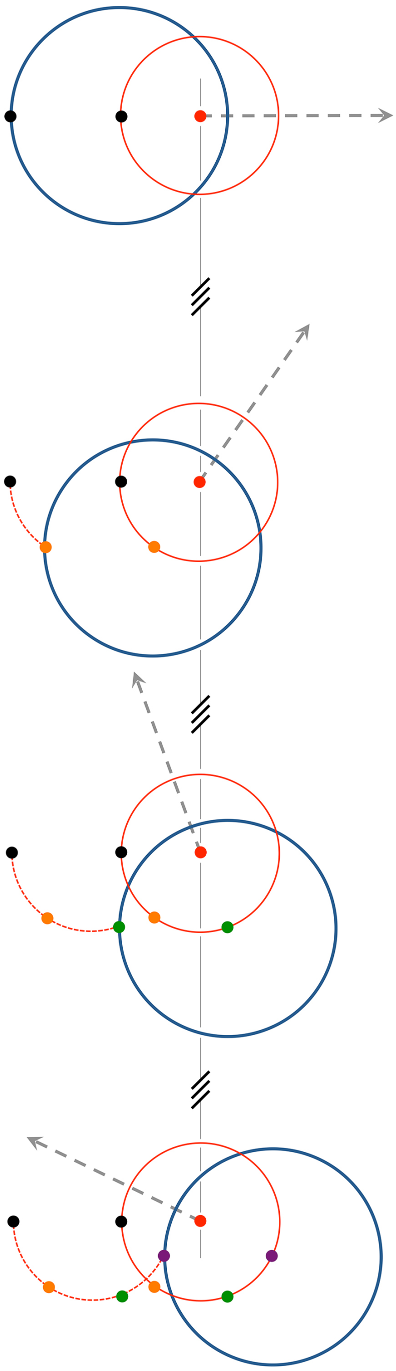

Figure 5: Illustration of the movement of the Earth (Earth's circumference in blue) around the Earth-Moon centre of mass (red dot), looking from the north, i.e., looking down onto the Earth-Moon system. The Earth's rotation around it's own rotation axis is ignored in the following (but will, obviously, be included later in the theory).

(1) Upper panel: The Moon is located to the right of the Earth, in the direction of the grey arrow. The black dots show the Earth's centre and a point on the central line, opposite to the direction of the Moon. The red circle shows the path the Earth's centre would follow if it rotates around the joint centre of mass.

(2) Second panel: Some time later, the Moon has moved counter-clockwise relative to the Earth. The Earth's centre (orange dot) is located on the circle around the joint centre of mass. Similarly, the point initially marked with the black dot on the opposite side of the Moon describes the same circular revolution as Earth's centre (the left-most yellow dot, with the path traced out by the movement shown by the red dotted arc).

(3) Third panel: Later, the Earth's centre (green dot) continue to follow the red, solid circle. The other point follows a similar circular trajectory (the second green dot and the red dotted arc).

(4) Lowermost panel: When the Moon is on the opposite side of the Earth, both of the two initial points are tracing out circles with a common radius $s = 0.73\,r$.

The resulting circular movement will continue as the Moon rotates around the Earth. In this way, all points on and within the solid Earth will describe circular trajectories with radius $s = 0.73\,r$. Note that the Earth does *not* rotate as a solid body around the Earth-Moon centre of mass. Rather, any point on Earth, like the left-most dot on the Earth's surface in all of the panels, have the same orientation with respect to a fixed star throughout the movement. This movement is commonly described as *revolution without rotation*.

can be fully understood with these assumptions, albeit accurate modelling of the Earth-Moon-Sun system requires taking into account these (relatively) slowly changing factors.

5.7 Net gravity on the Earth caused by the Moon

The distance between the centres of the Earth and the Moon vary slowly during the lunar month, but it can be treated as constant for the purposes considered here. If so, the magnitude of the outward-directed centrifugal acceleration at the centre of the Earth has to exactly balance the magnitude of the Moon's gravitational pull at the Earth's centre. This means that

$$a_\omega = a_O \tag{15}$$

or

$$\omega^2 \, s = G \, \frac{m_l}{R_l^2} \tag{16}$$

At the point P at the surface of the Earth, the gravity caused by the Moon will vary according to the distance to the Moon, with the largest gravitational pull for the points closest to the Moon. This pull is directed along the $\mathbf{q}$-vector. At the same time, the centrifugal acceleration in point P is directed opposite to the $\mathbf{R}$-vector. The net acceleration felt at P can then be expressed as

$$\mathbf{a} = G \, \frac{m_l}{q^2} \, \frac{\mathbf{q}_l}{q_l} - \omega^2 \, s \, \frac{\mathbf{R}_l}{R_l} \tag{17}$$

or, by means of (16),

$$\mathbf{a} = G \, m_l \left(\frac{\mathbf{q}_l}{q_l^3} - \frac{\mathbf{R}_l}{R_l^3} \right) \tag{18}$$

5.8 Introducing the zenith angle and simplifying

The acceleration (18) can be expressed in terms of $\mathbf{r}$, $\mathbf{R}$ and ϕ. From Fig. 4, the law of cosines gives (dropping the subscript l for the time being)

$$q^2 = R^2 + r^2 - 2 \, R \, r \cos\phi \tag{19}$$

or

$$\begin{aligned} q &= \left(R^2 + r^2 - 2 \, R \, r \cos\phi\right)^{1/2} \\ &= R \left(1 - 2\frac{r}{R} \cos\phi + \frac{r^2}{R^2}\right)^{1/2} \\ &\approx R \left(1 - 2\frac{r}{R} \cos\phi\right)^{1/2} \\ &\approx R \left(1 - \frac{r}{R} \cos\phi\right) \end{aligned} \tag{20}$$

Here the smallness of r/R (appendix A) has been used in the second last expression, and the binomial theorem for rational exponents (614) has been used in the last equality. Consequently,

$$\frac{1}{q^3} = \frac{1}{R^3} \left(1 - \frac{r}{R} \cos\phi\right)^{-3} \approx \frac{1}{R^3} \left(1 + 3\frac{r}{R} \cos\phi\right) \tag{21}$$

and

$$\mathbf{a} = \frac{G \, m_l}{R^3} \left[\left(1 + 3\frac{r}{R} \cos\phi\right) \mathbf{q} - \mathbf{R} \right] \tag{22}$$

The vector sum (2) and the smallness of r/R, and reintroducing the subscript l, give

$$\mathbf{a} = \frac{G\, m_l\, r}{R_l^3} \left(3 \cos\phi \frac{\mathbf{R}_l}{R_l} - \frac{\mathbf{r}}{r} \right) \tag{23}$$

Substituting G with Earth's gravitational acceleration g from (12) yields

$$\mathbf{a} = g\, \frac{m_l}{m_e} \left(\frac{r}{R_l} \right)^3 \left(3 \cos\phi \frac{\mathbf{R}_l}{R_l} - \frac{\mathbf{r}}{r} \right) \tag{24}$$

With the mass and distance ratios given in appendix A, it follows that **/p. 32/**

$$\boxed{|\mathbf{a}| \approx 10^{-6}\, g} \tag{25}$$

The tidal acceleration on Earth caused by the presence of the Moon is therefore very small. By decomposing the tidal acceleration into one component in the direction of $\mathbf{r}$ and one component tangential to the surface of the Earth, it follows that the former indeed can be ignored. The latter has no counterpart and it is this component that gives rise to the tides.

Note also that gravity is a *body force* (also called a *volume force*), implying that the tidal force acts throughout the entire water body, explaining that an apparently small acceleration actually gives rise to prominent variations throughout the entire ocean column.

5.9 Radial and tangential components of the tidal acceleration

5.9.1 Geometric approach

The component of the tidal acceleration in the direction of $\mathbf{R}$ follows directly from (24)

$$a_R = 3\, g\, \frac{m_l}{m_e} \left(\frac{r}{R_l} \right)^3 \cos\phi \tag{26}$$

$\mathbf{a}_R$ can be decomposed in the radial and horizontal directions based on Fig. 6.

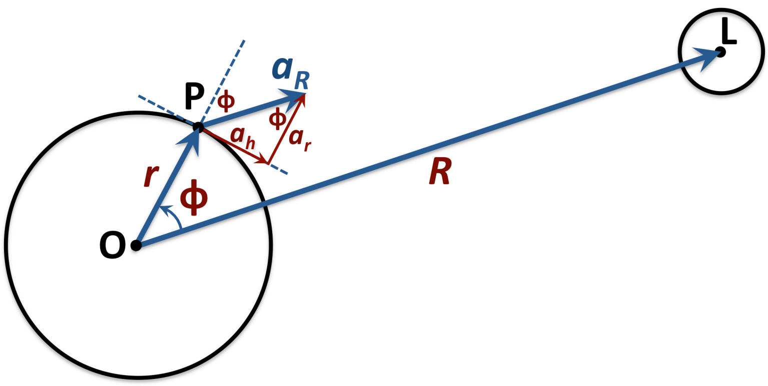

Figure 6: As Fig. 4, but for decomposing a_R (see expression 26) in the outward radial direction ($\mathbf{a}_r \parallel \mathbf{r}$) and in the horizontal direction on Earth, pointing towards the centre line ($\mathbf{a}_h$).

It follows from the figure that the horizontal component of a_R, in the direction towards the centre line, is

$$a_h = 3\,g\,\frac{m_l}{m_e}\left(\frac{r}{R_l}\right)^3 \sin\phi\cos\phi \tag{27}$$

Since

$$\sin\phi\cos\phi = \frac{1}{2}\sin 2\phi \tag{28}$$

(see 628), expression (27) can alternatively be expressed as

$$a_h = \frac{3}{2}\,g\,\frac{m_l}{m_e}\left(\frac{r}{R_l}\right)^3 \sin 2\phi \tag{29}$$

Likewise, the radial component of a_R is

$$3\,g\,\frac{m_l}{m_e}\left(\frac{r}{R_l}\right)^3 \cos^2\phi \tag{30}$$

The latter, together with the radial component of (24), adds up to

$$a_r = g\,\frac{m_l}{m_e}\left(\frac{r}{R_l}\right)^3 (3\cos^2\phi - 1) \tag{31}$$

a_r is parallel to $\mathbf{r}$ in Fig. 4.

5.9.2 Using the definition of the dot and cross products

Alternatively, a_r can be obtained by taking the dot product of $\mathbf{a}$ and the unit vector $\mathbf{r}/r$ (see the geometic configuration in the left panel in Fig. 9):

$$a_r = \mathbf{a}\cdot\frac{\mathbf{r}}{r} = g\frac{m_l}{m_e}\left(\frac{r}{R_l}\right)^3\left(3\cos\phi\frac{\mathbf{R}_l\cdot\mathbf{r}}{R_l\,r} - \frac{\mathbf{r}\cdot\mathbf{r}}{r^2}\right) \tag{32}$$

The definition of the dot product gives

$$\mathbf{R}_l\cdot\mathbf{r} = R_l\,r\cos\phi \quad\text{and}\quad \mathbf{r}\cdot\mathbf{r} = r^2 \tag{33}$$

so (32) becomes

$$a_r = g\,\frac{m_l}{m_e}\left(\frac{r}{R_l}\right)^3 \left(3\cos^2\phi - 1\right) \tag{34}$$

which is identical to (31).

In a similar manner, the cross product of $\mathbf{a}$ with the unit vector $\mathbf{r}/r$ gives the horizontal component of $\mathbf{a}$:

$$a_h = \left|\mathbf{a}\times\frac{\mathbf{r}}{r}\right| = g\,\frac{m_l}{m_e}\left(\frac{r}{R_l}\right)^3\left(3\,\frac{|\mathbf{R}_l\times\mathbf{r}|}{R_l\,r}\cos\phi - \frac{|\mathbf{r}\times\mathbf{r}|}{r^2}\right) \tag{35}$$

The definition of the cross product gives

$$|\mathbf{R}_l\times\mathbf{r}| = R_l\,r\sin\phi \quad\text{and}\quad \mathbf{r}\times\mathbf{r} = 0 \tag{36}$$

so

$$a_h = 3\,g\,\frac{m_l}{m_e}\left(\frac{r}{R_l}\right)^3 \sin\phi\cos\phi \tag{37}$$

5.9.3 The radial component can be ignored; the tangential component is important

Since the gravity on Earth's surface caused by the Earth's mass g is orders of magnitude larger than the gravitational acceleration in the radial direction (see 25), a_r can safely be neglected compared to g. Since g has no counterpart to a_h, this acceleration component – aligned tangential to the surface of the Earth – cannot be ignored. It is a_h that gives rise to the tides on Earth.

Note that a_h only varies with ϕ, the zenith or central angle in Fig. 4. ϕ depends on the latitude and longitude of the position P, as well as the declination angle of the Moon (and the Sun). Introduction of these geometric factors are presented in Sec. 7.

The horizontal component of the tidal acceleration or tidal force – which is the actual tidal acceleration or force – is often named the *tractive* acceleration or force.

5.10 Surface elevation caused by the Moon

The tidal acceleration a_h will give rise to changes in the sea level. This again leads to a pressure gradient or, alternatively, a pressure force. The pressure force per unit mass (i.e., acceleration) is, per definition (M & P (2008), eqn. 6.6),

$$-\frac{1}{\rho}\nabla p \tag{38}$$

Here p is pressure and ρ is density.

For a fluid with approximately uniform (constant) density, the hydrostatic approximation gives

$$p = \rho\, g\, \overline{\zeta} \tag{39}$$

Here $\overline{\zeta}$ (m) is the free ocean surface, i.e., the elevation relative to a flat ocean, caused by Earth's gravity and the tidal force. This means that the pressure force per unit mass can be expressed as

$$-g\,\nabla\overline{\zeta} \tag{40}$$

It follows then from Newton's second law, $\mathbf{F} = m\,\mathbf{a}$, expressed in the form $\mathbf{F}/m = \mathbf{a}$, that

$$-g\nabla_h\,\overline{\zeta} = a_h \tag{41}$$

where ∇_h denotes the gradient operator tangential to the Earth's surface.

It is convenient to express the differential equation (41) in terms of the spherical coordinate system to the right in Fig. 34. In this case the z axis of the spherical coordinate system is oriented in the direction of the centre line $\mathbf{R}$ in Fig. 4. In this system, only the $\mathbf{e}_\phi$ component (see 503) has a contribution (since $r =$ konst. and there is symmetry in the Ψ-direction), so (41) becomes

$$-\frac{g}{r}\frac{\partial\overline{\zeta}}{\partial\phi} = a_h \tag{42}$$

or, by means of (29),

$$\frac{\partial\overline{\zeta}}{\partial\phi} + \frac{3}{2}\frac{m_l}{m_e}\frac{r^4}{R^3}\sin 2\phi = 0 \tag{43}$$

Integration over ϕ, using

$$\int \sin 2\phi \, d\phi = -\frac{1}{2} \cos 2\phi \tag{44}$$

(see 634) gives the Equilibrium Tide's surface elevation

$$\overline{\zeta} = \frac{3}{4} \frac{m_l}{m_e} \frac{r^4}{R^3} \cos 2\phi + C \tag{45}$$

where C is an integration constant.

5.10.1 Determining the integration constant for the Equilibrium Tide surface elevation

Conservation of water volume requires that the surface elevation anomaly $\overline{\zeta}$ must vanish when integrated over the sphere. This constraint can be used to determine the integration constant C by integrating (45) over a sphere with constant radius r by means of the zenith-angle coordinate system in the right panel of Fig. 34.

The first term on the right-hand-side of (45) is a constant multiplied with the factor $\cos 2\phi$. Integration over the sphere of the given term therefore corresponds to solving the double-integral (similarly to 501)

$$r^2 \int_{\Psi=0}^{2\pi} d\Psi \int_{\phi=0}^{\pi} \cos 2\phi \sin \phi \, d\phi \tag{46}$$

The integral involving ϕ can be solved by using the identity (see 635):

$$\int \cos 2x \sin x \, dx = \frac{\cos x}{2} - \frac{\cos 3x}{6} + C^* \tag{47}$$

where C^* is an integration constant. Expression (46) then becomes

$$r^2 \, 2\pi \left[\frac{\cos \phi}{2} - \frac{\cos 3\phi}{6} \right]_0^{\pi} = -r^2 \, \frac{4\pi}{3} \tag{48}$$

Thus, integration of (45) over a sphere with constant radius r leads to

$$\frac{3}{4} \frac{m_l}{m_e} \frac{r^4}{R^3} \left(-r^2 \frac{4\pi}{3} \right) + 4 \, r^2 \, \pi C = 0 \tag{49}$$

Consequently,

$$C = \frac{1}{4} \frac{m_l}{m_e} \frac{r^4}{R^3} \tag{50}$$

The Equilibrium Tide is therefore governed by the expression

$$\overline{\zeta} = \frac{1}{4} \frac{m_l}{m_e} \frac{r^4}{R^3} \left(3 \cos 2\phi + 1 \right) \tag{51}$$

Alternatively, from identity 624,

$$\cos 2\phi = 2 \cos^2 \phi - 1 \tag{52}$$

the Equilibrium Tide can be put in the commonly used form

$$\overline{\zeta} = \frac{3}{2} \frac{m_l}{m_e} \frac{r^4}{R^3} \left(\cos^2 \phi - \frac{1}{3} \right) \tag{53}$$

6 The combined surface elevation caused by the Moon and the Sun

Expression (53) is also valid for the Sun. Thus, the total surface tidal elevation $\overline{\zeta}$, with contributions from the Moon (subscript l) and the Sun (subscript s), becomes

$$\begin{aligned}\overline{\zeta} &= \frac{3}{2}\frac{m_l}{m_e}\frac{r^4}{R_l^3}\left(\cos^2\phi_l - \frac{1}{3}\right) + \frac{3}{2}\frac{m_s}{m_e}\frac{r^4}{R_s^3}\left(\cos^2\phi_s - \frac{1}{3}\right) \\ &= \frac{3}{2}r\left[\frac{m_l}{m_e}\left(\frac{r}{R_l}\right)^3\left(\cos^2\phi_l - \frac{1}{3}\right) + \frac{m_s}{m_e}\left(\frac{r}{R_s}\right)^3\left(\cos^2\phi_s - \frac{1}{3}\right)\right] \qquad (54)\end{aligned}$$

6.1 Amplitudes

It follows from the above expression that the lunar and solar tides have maximum amplitudes for $\phi_l = \phi_s = 0$, implying that point P is located on the centre line:

$$\frac{m_l}{m_e}\frac{r^4}{R_l^3} \quad \text{and} \quad \frac{m_s}{m_e}\frac{r^4}{R_s^3} \qquad (55)$$

With the values from appendix A, the numerical values of the above factors are 0.36 m (or 0.3585 m with four decimals) for the lunar tide and 0.16 m (0.1646 m) for the solar tide. The ratio between the two contributions are thus (for $\phi_l = \phi_s = 0$) **/p. 44/**

$$\boxed{\frac{\overline{\zeta}_l}{\overline{\zeta}_s} = 2.18 \quad \text{or} \quad \frac{\overline{\zeta}_s}{\overline{\zeta}_l} = 0.46} \qquad (56)$$

The lunar tide is therefore about twice as large as the solar tide.

The dependency of ζ relative to the centre-line, given by $\phi_{l,s}$ in Eq. (54), is shown in Fig. 7. Maximum value is found on the centre line with values given in the previous paragraph, minimum

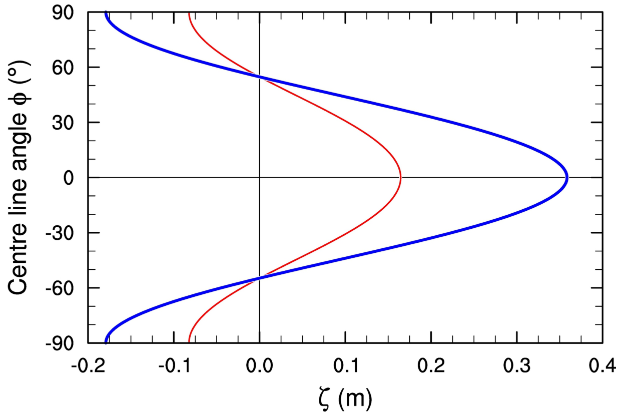

Figure 7: The magnitude of the equilibrium tide ζ (m) for angles $\phi_{l,s}$ (°) for the Moon (blue curve) and the Sun (red curve).

values are -0.18 m and -0.08 m at $\phi_{l,s} \pm 90°$, respectively, and $\zeta = 0$ for $\phi_{l,s} = 54.7°$.

The sum of the amplitude of the lunar and solar tides of about 0.5 m can be viewed as typical for tidal variations in the open ocean, but are, in general, lower than the observed tidal amplitude along the coasts. At some locations, the tidal amplitude may in fact exceed 10 m. The main reasons for the (localised) very large tidal amplitudes are the influence of the bathymetry, the detailed geometry of the coasts, and local eigen-modes.

6.1.1 Note on the magnitude of the lunar and solar tides

It follows from the previous section that the lunar tide is about twice as large as the solar tide (expression 56). Contrary to this, the gravitational force given by Newton's law of universal gravitation, expression (6), gives that gravitational pull from the Sun is about 180 times larger than the gravitational lunar pull.

This seemingly paradox can be explained by that the effective gravitational pull scales as $1/R^3$ (e.g., expression 29 and 53), whereas the gravitational force scales as $1/R^2$ (expression 6). Here R denotes the Earth–Moon, or Earth-Sun, distance. Since the Earth–Moon distance is much smaller than the Earth–Sun distance, the $1/R^3$-factor favour the tidal contribution from the Moon.

Alternatively, it is not the *magnitude* of the gravitational pull that determines the ocean tide, but the *spatial variations* of the gravitational pull. The former scales as $1/R^2$ (expression 6), whereas the latter scales as the derivative of $1/R^2$ with respect to R, or $1/R^3$ (e.g., expression 29 and 53).

The above argument illustrates the importance of understanding the physics of a given problem. In this case variations in the gravitational pull as experienced in point P on the Earth's surface, rather than the magnitude of the lunar or solar gravitational pull.

Furthermore, differences in the distance between any point P on Earth and the Moon or the Sun are given by the center line angle ϕ, explaining why the equilibrium tide can be expressed by means of the single variable ϕ in expression (54). See also the lower part of Fig. 4, illustrating identical gravitational pull along the red circle spun out by $\phi = \text{const}$.

6.2 Relative contribution from the Sun and the Solar planets

The maximum gravitational attraction on the Earth from any planet (subscript i) relative to that of the Moon is, from (55), given by the relationship

$$\frac{m_i}{m_l}\frac{R_l^3}{R_i^3} \tag{57}$$

With approximate values of any planet's mass m_i and the distance from the Earth's centre R_i, the relative magnitudes are as shown in Fig. 8. It follows that the Venus is the third most important planet, after the Moon and the Sun, with a contribution approximately $1/19{,}000$ of that of the Moon. Thus, the Moon and the Sun are, by far, the most important contributors to the Earth's ocean tide.

6.3 Tidal potential

In the case of the gravitational force (6), we get

$$G\frac{m_1 m_2}{l^2} = -\nabla\phi_t \tag{58}$$

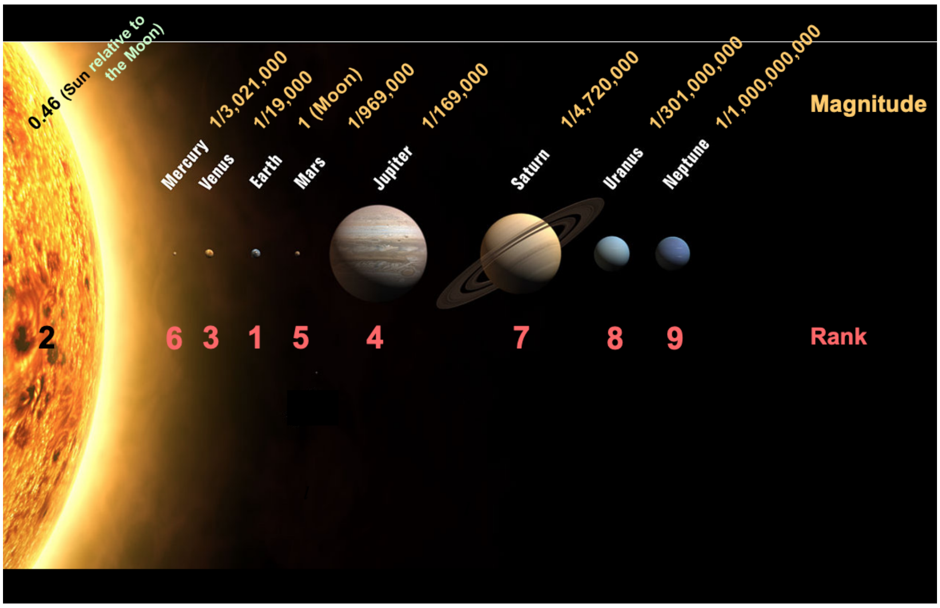

Figure 8: Approximate magnitude of the gravitational attraction on the Earth from the Sun and the Solar planets relative to that of the Moon, and the corresponding rank. Background illustration from `https://en.wikipedia.org/wiki/Solar_System`.

(with subscript t for *tide*). Integration over l

$$\int G\frac{m_1 m_2}{l^2}\,dl = -\int \frac{\partial \phi_t}{\partial l}\,dl \tag{59}$$

gives

$$G\frac{m_1 m_2}{l} = -\phi_t + C \tag{60}$$

where C is an integration constant.

The gravitational pull should vanish for infinitely large values of l. The condition $\phi(l \to \infty) = 0$ means that $C = 0$, so

$$\phi_t = -G\frac{m_1 m_2}{l} \tag{61}$$

ϕ_t in (61) represents force times length, or work, with units N m = kg m^2 s^{-2}. ϕ_t in (61) is called the *tidal potential* or the *gravitational potential energy*.

6.4 Gravitational potential

The gravitational potential energy per unit mass is called the *gravitational potential*. With $\phi_t^* = \phi_t/m_1$, the gravitational potential becomes

$$\phi_t^* = -G\frac{m_2}{l} \tag{62}$$

7 Introducing latitude, hour and declination angles

Up to now, no specific geographic reference has been given beyond the zenith angle shown in the left panel of Fig. 9. To determine the tide at any point P on the Earth's surface relative the the Moon or the Sun (or other celestial bodies), it is *convenient* to introduce three angles: The northern latitude ϕ_P of P, the hour angle C_P between the meridians going through the sub-lunar point[3] V and P (and thus playing the role of longitude), and the declination angle d of the celestial body (d_l for the Moon and d_s for the Sun), see the right panel in Fig. 9 for a depiction of the three angles.

7.1 The three leading lunar tidal components

We start by considering the two-body Earth-Moon system as illustrated in Fig. 9.

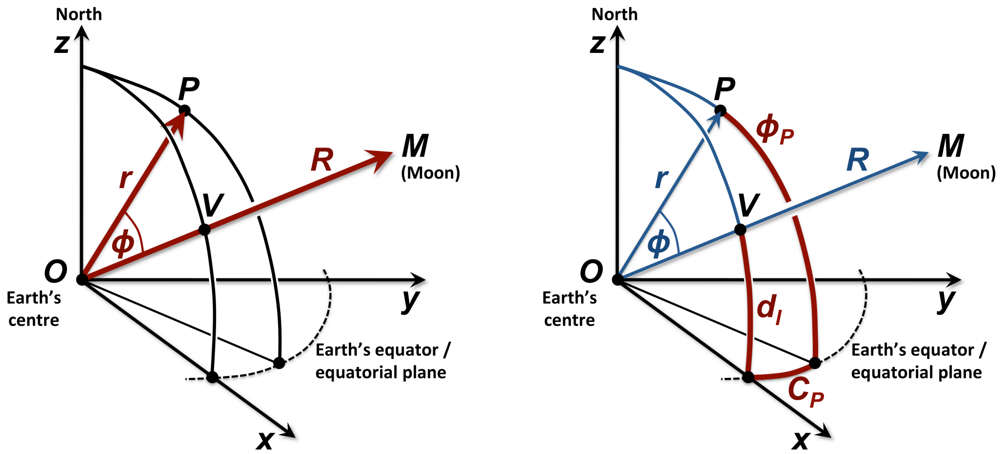

Figure 9: The tidal equilibrium configuration defined by $\mathbf{R}$, $\mathbf{r}$ and ϕ (left), and the new angles ϕ_P, C_P of and d_l (right). x, y, z denotes the Cartesian coordinates with x, y spanning the Earth's equatorial plane and z pointing northward. The x-axis is aligned along the projection of the centre line (or vector $\mathbf{R}$) onto the equatorial plane. O is the centre of the Earth, M is the Moon, and P is an arbitrary point on the surface of the Earth with latitude ϕ_P measured from the equatorial plane. The sub-lunar point V is where the centre line crosses the surface of the Earth (or the point on the surface of the Earth under the Moon) with the declination angle d_l. The hour angle C_P is the angle on the equatorial plane between the x-axis and the longitude of point P or, which is the same, the angle between the meridians running through V and P.

The two position vectors $\mathbf{r}$ and $\mathbf{R}$ can be expressed in terms of the angles ϕ_P, C_P and d_l. The projection of $\mathbf{R}$ onto the x- and z-axes give $R \cos d_l$ and $R \sin d_l$, respectively, see left panel of Fig. 10. Thus,

$$\mathbf{R} = R\,(\cos d_l, 0, \sin d_l) \tag{63}$$

Likewise, the projection of $\mathbf{r}$ onto the equatorial plane gives a vector of length $r \cos\varphi$ (right panel of Fig. 10). Decomposing this vector onto the x and y axes gives

$$\mathbf{r} = r\,(\cos\phi_P \cos C_P, \cos\phi_P \sin C_P, \sin\phi_P) \tag{64}$$

[3] *fotpunkt under månen*

Expression (53) states that the surface elevation of the Equilibrium Tide $\overline{\zeta}$ is proportional to the factor $\cos^2\phi - 1/3$. The task is therefore to express $\cos\phi$ in terms of ϕ_P, C_P and d_l. This can be readily done by means of the dot product

$$\begin{aligned}\mathbf{r}\cdot\mathbf{R} &= rR\,\cos\phi \\ &= rR\,(\cos\phi_P\cos d_l\cos C_P + \sin\phi_P\sin d_l)\end{aligned} \quad (65)$$

where the first expression is simply the definition of the dot product, whereas 63 and 64 have been used to explicitly evaluate the dot product in the second expression. From the above relationship it follows that

$$\boxed{\cos\phi = \sin\phi_P\sin d_l + \cos\phi_P\cos d_l\cos C_P} \quad (66)$$

which is the last expression on p. 41 in Pugh & Woodworth (2014). /p. 41/

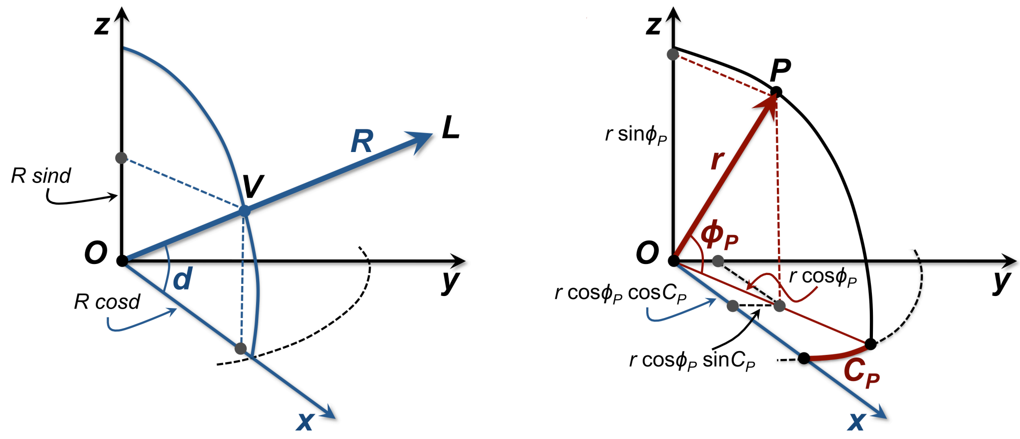

Figure 10: As Fig. 9, illustrating of the decomposition of $\mathbf{R}$ (left panel) and $\mathbf{r}$ (right panel) onto the Cartesian x, y, z-system.

Thus,

$$\cos^2\phi - \frac{1}{3} = \cos^2\phi_P\cos^2 d_l\cos^2 C_P + 2\cos\phi_P\cos d_l\cos C_P\sin\phi_P\sin d_l + \sin^2\phi_P\sin^2 d_l - \frac{1}{3} \quad (67)$$

The identity

$$\sin(2\,a) = 2\,\sin a\cos a \quad (68)$$

can be applied to the ϕ_P- and d_l-factors in the second term on the right hand side of (67), resulting in

$$\cos^2\phi - \frac{1}{3} = \cos^2\phi_P\cos^2 d_l\cos^2 C_P + \frac{1}{2}\sin(2\,\phi_P)\sin(2\,d_l)\cos C_P + \sin^2\phi_P\sin^2 d_l - \frac{1}{3} \quad (69)$$

The right hand side of (69) contains terms with variations on mainly three different time scales. Keeping the latitude of the position P fixed so $\phi_P =$ const., the first term on the right hand side of (69) has temporal variattions mainly governed by $\cos^2 C_P$. This term gives rise to the semi-diurnal lunar tide. The second term on the right hand side of (69) varies mainly because of $\cos C_P$, describing the diurnal lunar tide. And finally, the third term vary with $\sin^2 d_l$, describing the tidal response to the near biweekly (or fortnightly) variations in the lunar declination.

The three time scales can be grouped by expressing

$$\cos^2\phi - \frac{1}{3} = \overline{\zeta}_0 + \overline{\zeta}_1 + \overline{\zeta}_2 \tag{70}$$

where

$$\overline{\zeta}_0 = \frac{3}{2}\left(\sin^2\phi_P - \frac{1}{3}\right)\left(\sin^2 d_l - \frac{1}{3}\right) \tag{71}$$

$$\overline{\zeta}_1 = \frac{1}{2}\sin(2\,\phi_P)\sin(2\,d_l)\cos C_P \tag{72}$$

$$\overline{\zeta}_2 = \frac{1}{2}\cos^2\phi_P\cos^2 d_l\cos(2\,C_P) \tag{73}$$

Here, $\overline{\zeta}_0$ describes the biweekly (fortnightly) variations (governed by $\sin^2 d_l$), $\overline{\zeta}_1$ describes the diurnal variations (governed by $\cos C_P$), and $\overline{\zeta}_2$ describes the semi-diurnal variations (governed by $\cos C_P$). The transformation of (69) into (70)–(73) is given in appendix C.

Expression (70) corresponds to the terms in square brackets in expression (3.10) in Pugh & Woodworth (2014). **/3.10/**

7.2 The Equilibrium Tide caused by the Moon

By combining (54) with expressions (70)–(73), the Equlibrium Tide caused by the Moon, $\overline{\zeta}_l$, can be put in the form (see expression 3.12 in Pugh & Woodworth, 2014): **/3.12/**

$$\boxed{\overline{\zeta}_l = r\,\frac{m_l}{m_e}\left[C_0(t)\left(\frac{3}{2}\sin^2\phi_P - \frac{1}{2}\right) + C_1(t)\sin 2\phi_P + C_2(t)\cos^2\phi_P\right]} \tag{74}$$

where

$$C_0(t) = \left(\frac{r}{R_l}\right)^3\left(\frac{3}{2}\sin^2 d_l - \frac{1}{2}\right) \tag{75}$$

$$C_1(t) = \left(\frac{r}{R_l}\right)^3\frac{3}{4}\sin 2d_l\cos C_P \tag{76}$$

$$C_2(t) = \left(\frac{r}{R_l}\right)^3\frac{3}{4}\cos^2 d_l\cos 2C_P \tag{77}$$

A similar expression is valid for the solar Equilibrium Tide.

7.3 Properties of the three leading lunar components

From (71)–(73), it follows that all components depend on the latitude ϕ_P. Secondly, the identities

$$\cos^2 d_l = \frac{1}{2}\left(1 + \cos 2\,d_l\right) \tag{78}$$

and

$$\sin^2 d_l = \frac{1}{2}\left(1 - \cos 2\,d_l\right) \tag{79}$$

show that all contributions depend on $\cos 2\,d_l$, implying that all contributions have a periodicity of $T_l/2 \sim 14$ d (see Sec. 8.1 for a discussion of T_l, but it is sufficient here to note that $T_l = 27.3216$ d). And thirdly, only $\overline{\zeta}_1$ and $\overline{\zeta}_2$ depend on the hour angle C_P.

Other characteristics are listed below.

7.3.1 Declination or nodal component, $\overline{\zeta}_0$

- $\overline{\zeta}_0$ vanishes for $\sin^2\phi_P = 1/3$, or for the latitudes $\phi_P = \pm 35°$
- The first parenthesis $(\sin^2\phi_P - 1/3)$ is thus positive for $|\phi_P| > 35°$ and negative for $|\phi_P| < 35°$
- The contribution from ϕ_P is largest at the poles ($\phi_P = \pm 90°$)
- Since the Moon's declination angle d_l varies between $-(18.5°$ to $28.5°)$ and $+(18.5°$ to $28.5°)$, see footnote [4], the second parenthesis $(\sin^2 d_l - 1/3)$ is always negative, with largest values when the Moon is high on sky
- In total, $\overline{\zeta}_0$ is positive for $|\phi_P| < 35°$ and negative at higher latitudes

7.3.2 Diurnal component, $\overline{\zeta}_1$

- The leading, temporal variability is governed by $\cos C_P$, with a period of 24 h 50 min (see Sec. 8.2)
- $\overline{\zeta}_1$ is positive when the point P and the Moon are on the same hemisphere, negative otherwise
- $\sin(2d_l) = 0$ twice a month, so $\overline{\zeta}_1$ vanishes when the Moon crosses the Earth's equatorial plane twice a month
- The d_l-modulation is largest when the Moon is highest on sky
- Largest signal at $\phi_P = \pm 45°$, vanishes at equator and the poles

7.3.3 Semi-diurnal component, $\overline{\zeta}_2$

- The leading, temporal variability is governed by $\cos 2\,C_P$, with a period of 12 h 25 min (see Sec. 8.2)
- $\overline{\zeta}_2 \geq 0$ for all values of ϕ_P and d_l
- d_l-modulation is maximum when the Moon crosses the equatorial plane twice a month ($d_l = 0$), and it decreases with increasing $|d_l|$; $\min(\cos^2 d_l) \approx 0.77$ for $d_l = \pm 28.5°$
- Largest signal at $\phi_P = 0$ (equator), vanishes at the poles

[4] Maximum declination angle d_l varies with the nodal period – or the regression period of the Moon's nodes – of 18.61 years, see table 2.

8 Sidereal and synodic periods

The solar system is characterised by cyclic variations, like Earth's day-night cycle due to Earth's rotation around it's own rotation axis, the close to monthly lunar phase as seen from the Earth, and the annual (seasonal) cycle as observed from the Earth. These periodicities need to be uniquely and accurately determined in order to quantify the detailed lunar and solar contributions to the tides.

The various periods can be determined from an observer on the (rotating) Earth, or as seen from a (stationary) fixed star. Seen from the Earth, the periodicity is named the *synodic period*[5] (from "synodus", Greek from "coming together"). Seen from a distant star, the period is known as the *sidereal period*[6] (from "sidus", or "star", in Latin).

8.1 The Moon's sidereal and synodic monthly periods

A full rotation of the Moon around the Earth as seen from a distant star has a sidereal period $T_l = 27.3216$ d (this is an observed, given value). The synodic period can be found by considering the Sun-Earth-Moon system seen from a fixed star from above (north), as illustrated in Fig. 11.

If the Earth were not rotating around the Sun, the Moon's rotation rate as seen from the Earth would equal

$$\omega_l \, T_l = 2\pi \tag{80}$$

In this case the Moon, initially located on the line between the Sun and the Earth (point A in Fig. 11), would be back on the same line after the period T_l .

However, since the Earth does rotate around the Sun with the rotation rate ω_{es} and a rotation period of $T_{es} = 365.2422$ d, the Moon will not be back at the initial position – seen from the Earth – after the sidereal period T_l. Actually, the Moon needs to move from point B to point C in Fig. 11 in order to be back at the initial position, i.e., back to the position where the Earth, the Moon and the Sun are all aligned. We therefore search for a period $T^* > T_l$ in order for the Moon to reach the position C in Fig. 11.

Based on (80), T^* can be derived from the expression

$$\omega_l \, T^* = 2\pi + \omega_{es} \, T^* \tag{81}$$

where the last term corresponds to the angle between A, the centre of the Sun and C or, which is equivalent, the angle between B, the centre of the Earth and C in Fig. 11 (both identified by the red, double-lined arcs). Division by the factor $2\pi \, T^*$ gives

$$\frac{1}{T_l} = \frac{1}{T^*} + \frac{1}{T_{es}} \tag{82}$$

For $T_l = 27.3216$ d and $T_{es} = 365.2422$ d, we get that the lunar synodic period

$$T^* = 29.5307 \text{ d} \tag{83}$$

Thus, the monthly lunar period seen from a distant star is 27.3216 d (the monthly sidereal lunar period), whereas it is 29.5307 d seen from the Earth (the monthly synodic lunar period).

[5] *synodisk periode*
[6] *siderisk periode*

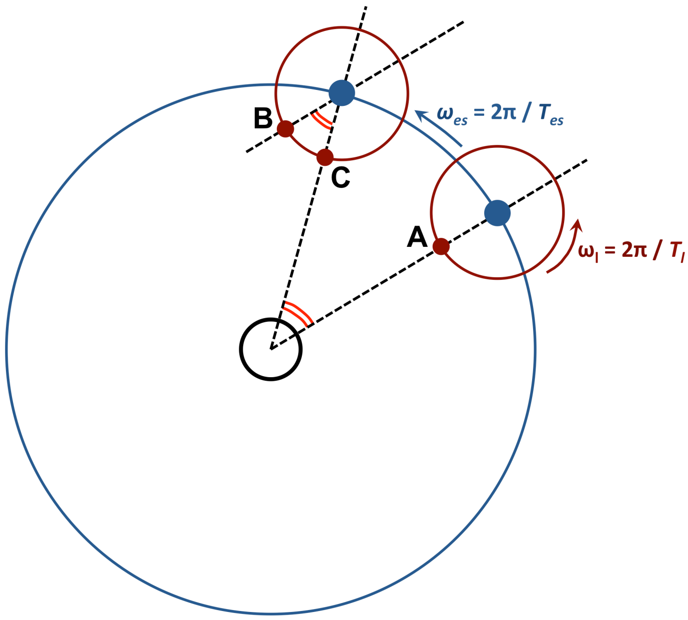

Figure 11: Illustration of the Sun (black circle in the centre), the Earth (solid blue dot, encircling the Sun along the blue circle), and the Moon (solid red dot, encircling the Earth along the red circle), viewed from the north. ω_l and T_l are the lunar rotation rate and period seen from a distant star, respectively. Similarly, ω_{es} and T_{es} are the Earth's rotation rate and rotation period around the Sun. Point A denotes the initial position of the Moon, in this case exactly on the centre line between the Earth and the Sun. Point B refers to the position of the Moon after the sidereal period T_l. Point C denotes the full lunar rotation as observed from the Earth, in this case when the Moon is back on the centre line between the Earth and the Sun. The rotation period for the latter is the synodic period.

8.2 The synodic lunar day

Following the logic from Sec. 8.1, the lunar day T^* seen from Earth – due to the Earth's rotation around it's own axis – will be somewhat longer than that seen from a fixed star. The situation is illustrated in Fig. 12.

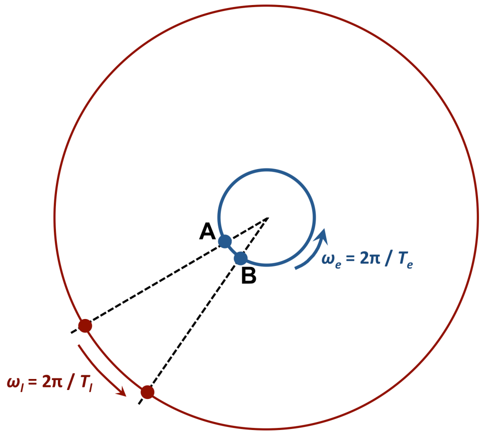

Figure 12: Illustration of the lunar day. The Moon is the filled red dot encircling the Earth along the red circle, viewed from above (north). The Earth is the open blue circle. If the Moon was fixed in space and if it was located above point A on the Earth's surface initially, the Moon would be located in the same position after one full rotation of the Earth ($T_e = 1$ d). However, during the period T_e, the Moon has moved with the rotation rate ω_l given by the lunar synodic month from Sec. 8.1. The Earth must therefore make some additional rotation, corresponding to moving point A to position B, before the Moon is again located in zenith relative to the initial point A.

One full rotation of the Earth can be expressed as

$$\omega_e T_e = 2\pi \tag{84}$$

where $T_e = 1$ d.

Taking into account the simultaneous rotation of the Moon around the Earth, gives

$$\omega_e T^* = 2\pi + \omega_l T^* \tag{85}$$

where T^* is the lunar day as seen from the Earth.

Division by $2\pi T^*$ gives

$$\frac{1}{T_e} = \frac{1}{T^*} + \frac{1}{T_l} \tag{86}$$

With $T_l = 29.5307$ d and $T_e = 1$ d, we obtain

$$T^* = 1.0351 \text{ d} \tag{87}$$

or

$$T^* = 24\,\text{hr}\ 50\,\text{min}\ 32.64\,\text{sec} \approx 24\,\text{hr}\ 50\,\text{min} \tag{88}$$

The lunar day observed from the Earth, the synodic lunar day, is therefore about 50 minutes longer than the solar day on the Earth.

8.3 The Earth's daily sidereal period

The daily period of the Earth seen from a fixed star is somewhat shorter than the mean solar day because of the movement of the Earth around the Sun. The situation is illustrated in Fig. 13.

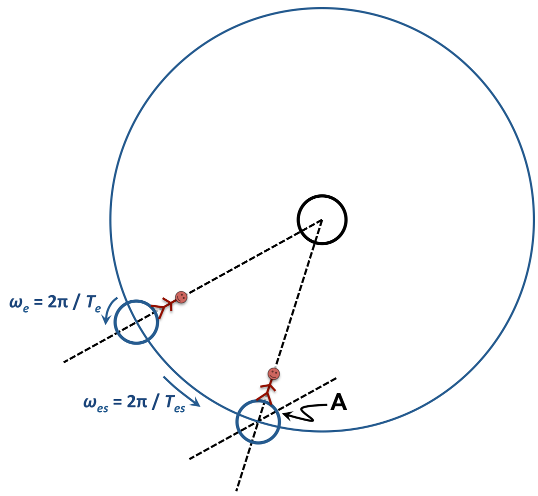

Figure 13: Illustration of the Earth's sidereal day. The Earth (blue circle) encircles the Sun (black circle), viewed from above (north). If the Earth did not rotate around the Sun, any point on the Earth's surface (illustrated with a person with the Sun in zenith) would be back at the same position relative to the Sun after 1 mean solar day, $T_e = 1$ d. However, during the time T_e, the Earth has moved with the rotation rate ω_{es} around the Sun. Therefore, seen from a fixed star, the Earth has made slightly more than one full rotation before the person in the figure has the Sun in zenith (note that the dashed line at point A runs parallel to the line going through the left-most person). For this reason, the sidereal day, i.e., time from the starting point until the the person is in position A, must be slightly *shorter* than T_e.

One full rotation of the Earth can be expressed as

$$\omega_e T_e = 2\pi \tag{89}$$

where $T_e = 1$ d.

Taking into account the simultaneous rotation of the Earth around the Sun, gives

$$\omega_e T^* = 2\pi - \omega_{es} T^* \tag{90}$$

where T^* is the Earth's sidereal day and the minus-sign is introduced since $T^* < T_e$.

Division by $2\pi T^*$ gives

$$\frac{1}{T_e} = \frac{1}{T^*} - \frac{1}{T_{es}} \tag{91}$$

With $T_{es} = 365.2422$ d (the lunar synodic month, from the previous section) and $T_e = 1$ d, we obtain

$$T^* = 0.9973 \text{ d} \tag{92}$$

or

$$T^* \approx 23\,\text{hr}\ 56\,\text{min} \tag{93}$$

Earth's rotation period observed from a fixed star – the Earth's daily sidereal period – is therefore about 4 minutes less than the mean solar day.

8.4 The synodic lunar day

In this case we consider rotation periods relative to a fixed star, for instance relative to the first point of Aries[7], ♈, see Fig. 14.

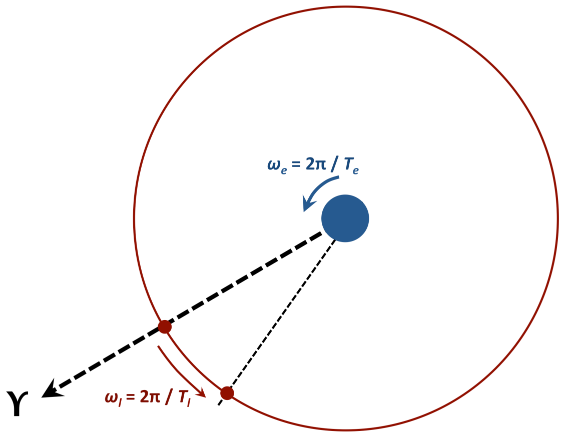

Figure 14: The Earth is the blue disc, rotating with a period $T_e = 0.9973$ d relative to a fixed star, exemplified with the first point of Aries ♈ (viewed from above, i.e., north). During the time T_e, the Moon (small red disc) rotates around the Earth with a rotation period $T_l = 27.3216$ d.

One full rotation of the Earth relative to ♈ is given by

$$\omega_e\,T_e = 2\pi \tag{94}$$

where $T_e = 0.9973$ d.

Taking into account the simultaneous rotation of the Moon around the Earth, again relative to a fixed star, leads to the relationship

$$\omega_e\,T^* = 2\pi + \omega_l\,T^* \tag{95}$$

where ω_l corresponds to the lunar period $T_l = 27.3216$ d.

Division by $2\pi\,T^*$ gives

$$\frac{1}{T_e} = \frac{1}{T^*} + \frac{1}{T_l} \tag{96}$$

Thus

$$T^* = 1.0351\ \text{d} \tag{97}$$

implying that the lunar day as seen from the Earth – the synodic lunar day – is 1.0351 d, in accordance to Sec. 8.2.

[7] *stjernetegnet Væren, eller vårpunktet*

8.5 Key periods and frequencies for the Moon-Earth-Sun system

Based on the above sections, a set of basic astronomical periods and frequencies can be defined to describe the Earth-Moon-Sun system. With msd denoting the *mean solar day* and msh denoting the *mean solar hour*, these are:

Quantity	Symbol	Unit
Period	T	msd or msh
Frequency	$f = 1/T$	1/msd or 1/msh
Angular speed	$\sigma = 360°/T$	deg/msh
Angular speed	$\omega = 2\pi/T$	rad/msh
Physical angles	$C_s, C_l, s, h, p, N', p'$	rad

Table 1: Overview of commonly used quantities in tidal analysis. The physical meaning of the seven physical angles are indicated in Table 2 and will be explicitly defined in the subsequent sections.

For historical and practical reasons that will become clear(er) in Sec. 9.7, the various periods and rotation rates are put in the form T_i, σ_i and ω_i, where the subscript i is an integer, with the exception of Earth's sidereal day that is denoted with the subscript s, see Table 2. **/Tab. 3.2/**

		Period		Frequency	Angular speed		Corresp. angle
		T		f	σ	ω	
			time unit	cycles per time unit	deg per msh	rad per msh	deg / rad
Mean synodic solar day	0	1.00	msd	1.00	15.0000	0.2618	C_s
Mean synodic lunar day	1	1.0351	msd	0.9661369	14.4921	0.2529	C_l
Lunar sidereal month	2	27.3216	msd	0.0366011	0.5490	0.0096	s
Tropical year	3	365.2422	msd	0.0027379	0.0411	0.0007	h
Moon's perigee	4	8.85	Jyr	0.0003093	0.0046		p
Regression of Moon's nodes	5	18.61	Jyr	0.0001471	0.0022		N'
Perihelion	6	20,942	Jyr	–	–		p'
Earth's sidereal day	s	0.9973	msd	1.0027973	15.0411	0.2625	–
Lunar synodic month	–	29.5307	msd	0.0338631	0.5079	0.0089	–

Table 2: Overview of the basic astronomical periods and frequencies. msd is the *mean solar day*, msh is the *mean solar hour* and Jyr is the Julian year (365.25 d). The second row gives the index of the different quantities, e.g., the period of the Moon's perigee is $T_4 = 8.85$ Jyr. Note 1: A tropical year is the time between the Sun's successive crossings of the first point of Aries ♈. Note 2: Because of the precession of the equinoxes `https://en.wikipedia.org/wiki/Axial_precession` (see *Axial precession* on Wikipedia), the seasonal cycle does not remain exactly synchronized with the position of the Earth in its orbit around the Sun. As a consequence, the tropical year is about 20 minutes shorter than the time it takes Earth to complete one full orbit around the Sun as measured with respect to the fixed stars (the sidereal year) `https://en.wikipedia.org/wiki/Tropical_year` (see *Tropical year* on Wikipedia). The basic astronomical periods are illustrated given in Part I.

With the definitions in Table 2, the relationship (90) is equivalent to

$$\omega_2 = \frac{2\pi}{T^*} - \omega_3 \qquad (98)$$

In expressin (90), the Earth's sidereal day is given by $T^* = 2\pi/\omega_s$, so $2\pi/T^* = \omega_s$. Consequently, /p. 48/

$$\boxed{\omega_s = \omega_0 + \omega_3 \quad \text{or} \quad \sigma_s = \sigma_0 + \sigma_3} \tag{99}$$

Likewise, from expression (95), /p. 48/

$$\boxed{\omega_s = \omega_1 + \omega_2 \quad \text{and} \quad \sigma_s = \sigma_1 + \sigma_2} \tag{100}$$

Therefore,

$$\boxed{\omega_1 + \omega_2 = \omega_0 + \omega_3 \quad \text{and} \quad \sigma_1 + \sigma_2 = \sigma_0 + \sigma_3} \tag{101}$$

8.6 The lunar and solar hour angles expressed by the sidereal time

The hour angle C_P in Fig. 9 is the angle between the meridians running through the sub-lunar point V and an arbitrary point P on the Earth's surface, denoted C_l. Similarly, the angle between the meridians running through the sub-solar point V and an arbitrary point P on the Earth's surface is C_s. Time t can be introduced with the convention that t is measured from Υ and is expressed relative to a fixed star, known as the *sidreal time*. The description in the following paragraphs represents the Earth-Moon system, but corresponding definitions hold for the Earth-Sun system.

Projected onto the Earth's equatorial plane, and by introducing the first point of Aries Υ and the Greenwich meridional, Fig. 9 looks like Fig. 15. It follows that the lunar hour angle C_l

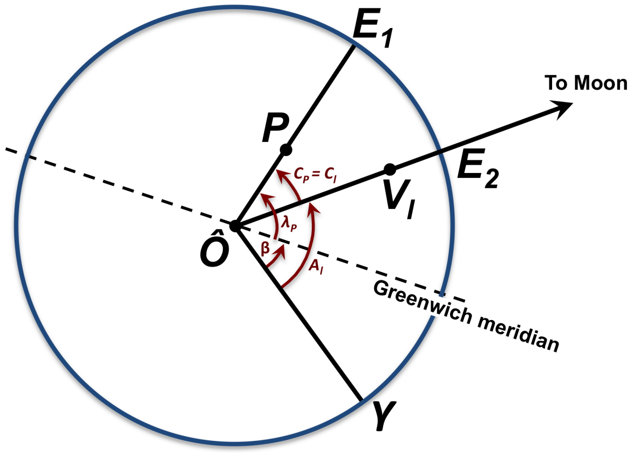

Figure 15: The celestial sphere, as shown in Fig. 9 for the Earth-Moon system, projected onto the Earth's equatorial plane. V_l is the sub-lunar point, P is an arbitrary point on the Earth's surface, and Υ is the first point of Aries. $C_P = C_l$ is the lunar hour angle, λ_P is the positive east longitude of P, β is the the *Right Ascension* of the Greenwich meridian, and A_l is the *lunar Right Ascension*. The angles $E_1\hat{O}\Upsilon = \lambda_P + \beta$ and $E_2\hat{O}\Upsilon = A_l$. A similar configuration is valid for the Earth-Sun system, denoted with the subscript s.

equals the difference between the angles $E_1\hat{O}\Upsilon$ and $E_2\hat{O}\Upsilon$:

$$C_l = E_1\hat{O}\Upsilon - E_2\hat{O}\Upsilon = \lambda_P + \beta - A_l \tag{102}$$

Here λ_P is the positive east longitude of P, β is the the *Right Ascension* of the Greenwich meridian, and A_l is the *lunar Right Ascension*.

The greenwich meridian, relative to a fixed star, moves with the speed $\omega_s = \omega_0 + \omega_3$, see Sec. 8.3. Consequently,

$$\beta = (\omega_0 + \omega_3)\, t \tag{103}$$

The *Greenwich Mean Time* is measured from the lower transit of the mean Sun, or from midnight at Greenwich. Therefore, with the Greenwich Mean Time used as time reference, which is commonly the case,

$$\beta = (\omega_0 + \omega_3)\, t - \pi \tag{104}$$

In summary, the lunar hour angle can be put in the form **/(3.20a)/**

$$\boxed{C_l = \lambda_P + (\omega_0 + \omega_3)\, t - \pi - A_l} \tag{105}$$

The solar hour angle C_s is similarly given by: **/(3.20b)/**

$$\boxed{C_s = \lambda_P + (\omega_0 + \omega_3)\, t - \pi - A_s} \tag{106}$$

In the above expression, A_s is the *solar Right Ascension*.

Note that expressions (105) and (106) include several time dependencies, all expressed relative to fixed stars: The fast-changing dependency is caused by Earth's diurnal rotation about it's own axis (the $\omega_0 + \omega_3$ term), whereas slower dependencies are given by the Moon's and the Sun's rotation around the Earth (C_l and C_s, respectively), the latter from a geocentric point of view. The explicit formulation of A_s can be derived from the Kepler's equations (Sec. 15) and is given by expression (298), with a corresponding expression for A_l. A very slow time dependency is given by the actual position of the first point of Aries. For most applications, the latter can be ignored.

In the special case that the point P is located on the Greenwich Meridian, $\lambda_P = 0$, and expressions (105) and (106) become **/p. 63/**

$$C_l = (\omega_0 + \omega_3)\, t - \pi - A_l \tag{107}$$

and

$$C_s = (\omega_0 + \omega_3)\, t - \pi - A_s \tag{108}$$

9 Decomposing the solar and lunar tides into a series of simple harmonic constituents

The total tide can be expressed as a sum of many harmonic contributions. Each of these contributions – called tidal component, tidal constituent or harmonic constituent – is represented by a simple, harmonic cosine function. A capital letter (sometimes a combination of capital and small letters), plus a numerical subscript, is used to designate each constituent. For instance, the semi-diurnal tidal contribution from the Moon and the Sun are named $\mathbf{M}_2$ and $\mathbf{S}_2$, respectively.

Each constituent is typically described by its speed (or frequency), expressed as degrees per mean solar hour (hereafter deg/msh) or radians per mean solar hour (rad/msh). The speed of a constituent is $360°/T$, where T (msh) is the period. For $\mathbf{S_2}$, the speed is then $360°/(12 \text{ msh}) = 30° / \text{msh}$.

9.1 Geometry of the Earth-Sun system

The Earth-Sun system, viewed from the Earth (i.e., a geocentric view), can be presented by means of the declination angle d_s, the ecliptic plane angle $\epsilon_s = \text{const} = 23°\, 27'$ and the longitude angle λ_s as shown in Fig. 16. From this figure, the following relationships are obtained from the three right-angled triangles mentioned in the figure caption

$$H = R \sin d_s \tag{109}$$

$$H = L \sin \epsilon_s \tag{110}$$

$$L = R \sin \lambda_s \tag{111}$$

The first two expressions give

$$\sin d_s = \frac{L}{R} \sin \epsilon_s \tag{112}$$

Combined with the third expression, one obtains (Pugh & Woodworth, p. 46) **/p. 46/**

$$\boxed{\sin d_s = \sin \lambda_s \, \sin \epsilon_s} \tag{113}$$

9.1.1 Angular speed of the Earth and the Sun

For the geocentric view in Fig. 16, the longitude angle λ_s describes the Sun's motion around the Earth with a speed given by the duration of the tropical year[8,9]

$$\frac{360°}{365.2422 \text{ msd}} = 0.0411 \, \frac{\text{deg}}{\text{msh}} \stackrel{\text{def}}{=} \sigma_3 \tag{114}$$

Similarly, C_s describes the Earth's rotation around it's own axis

$$\frac{360°}{24 \text{ msh}} = 15.0000 \, \frac{\text{deg}}{\text{msh}} \stackrel{\text{def}}{=} \sigma_0 \tag{115}$$

[8] *tropisk år*

[9] "tropical" comes from the Greek *tropicos* meaning "turn", denoting the turning of the Sun's seasonal motion as seen from the Earth, i.e., the time between the Sun's successive crossings of the first point of Aries (from ♈ to ♈), see Fig. 15. From `https://en.wikipedia.org/wiki/Tropical_year`: "Because of the precession of the equinoxes, the seasonal cycle does not remain exactly synchronised with the position of the Earth in its orbit around the Sun. As a consequence, the tropical year is about 20 minutes shorter than the time it takes Earth to complete one full orbit around the Sun as measured with respect to the fixed stars (the sidereal year)".

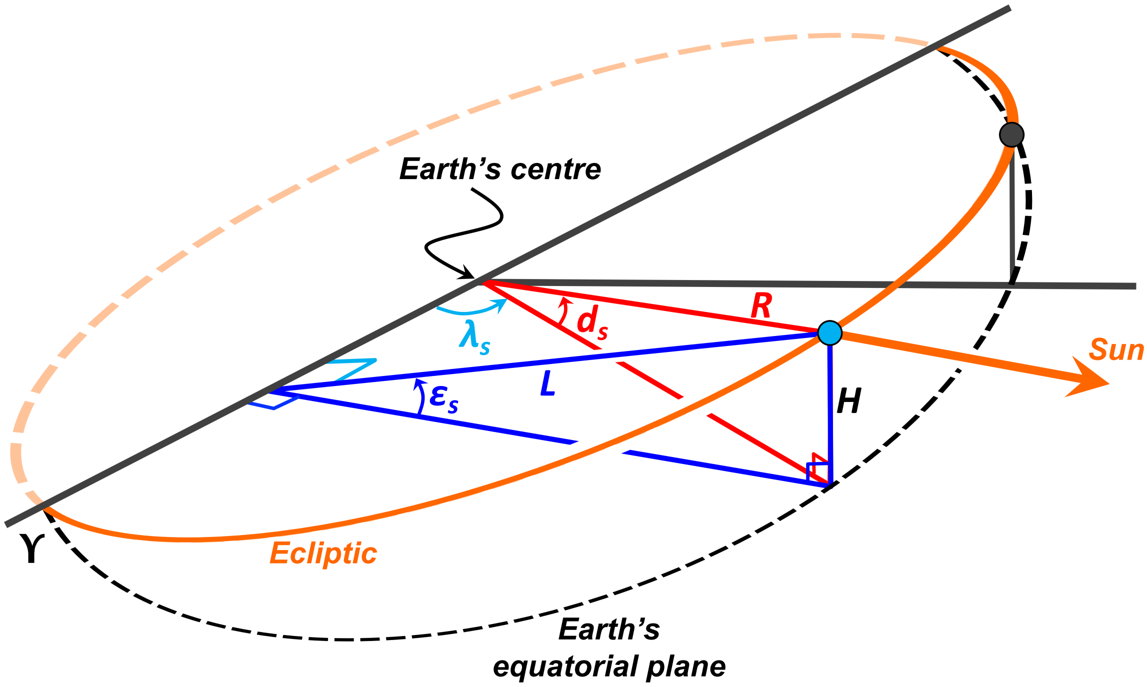

Figure 16: A geocentric view of the Earth-Solar system. $\mathbf{R}$ is the solar position vector, pointing from the Earth's centre to the centre of the Sun, with a declination angle d_s above Earth's equatorial plane. The Sun's path on sky follows the ecliptic, with ϵ_s denoting the (approximately) constant angle between Earth's equatorial plane and the ecliptic ($\epsilon_s = 23^\circ\, 27' = 23.5^\circ$). The third angle λ_s denotes the Sun's eastward directed longitudinal angle with respect to the vernal equinox Υ (the point where the ascending Sun crosses the equatorial plane). The vernal equinox Υ is also known as the first point of Aries. Note the three right-angled triangles in the figure; one in red, one in dark blue, and one in light blue confined by the line from the Earth's centre towards Υ and the sub-solar point (light blue dot). For the latter, the angle at the Earth's centre is the longitude λ_s. The triangle describing the angle between the Earth's equatorial plane and the ecliptic, the deep blue triangle, slides perpendicularly along the line between the vernal and autumnal equinoxes.

The angles λ_s and C_s, expressed in degrees from Υ can thus be written as

$$\lambda_s = \sigma_3\, t \quad \text{and} \quad C_s = \sigma_0\, t \tag{116}$$

where t is time in msh.

In order to split *products* of trigonometric functions into a sum or a difference of *individual* trigonometric functions (by means of the trigonometric identities 625–627, as demonstrated in the next section), we want to express λ_s and C_s with a *shared* frequency.

Since $\sigma_s = \sigma_0 + \sigma_3$, see expression (99), the two angels can be written, in degrees, as

$$\lambda_s = \sigma_3\, t \quad \text{and} \quad C_s = (\sigma_s - \sigma_3)\, t \tag{117}$$

Alternatively, λ_s and C_s can be expressed in terms of radians:

$$\lambda_s = \omega_3\, t \quad \text{and} \quad C_s = (\omega_s - \omega_3)\, t \tag{118}$$

In the above cases, λ_s and C_s have σ_3 (or ω_3) as the shared frequency. In the following, we express the temporal evolution of λ_s and C_s by means of ω_x, but σ_x can be used equally well (here the subscript x denotes any subscript).

9.1.2 Temporal change of the geographical latitude point C_P

The temporal change of the geographical point C_P in expressions (72) and (73), relative to the Sun, is governed by C_s. In the following, we include the temporal dependency of C_P by substituting C_P by $C_s = \sigma_0\, t = (\sigma_s - \sigma_3)\, t$ in expressions (72) and (73).

9.1.3 Expressing $\sin 2d_s$ as $\sin d_s$

Both $\overline{\zeta}_1$ (72) and $\overline{\zeta}_2$ (73) include a harmonic term with argument $2\,d_s$. We only have an expression for $\sin d_s$ (equation 113), not for $\sin 2d_s$. The latter term can, however, be expressed by means of $\sin d_s$ in the following way

$$\sin\, 2\,d_s \overset{(628)}{=} 2 \sin d_s \cos d_s \overset{(618)}{=} 2\, \sin d_s \sqrt{1 - \sin^2\, d_s} \tag{119}$$

The declination angle $|d_s| < 23.5^\circ$, so $|\sin^2 d_s|$ is a rather small factor. The first order approximation of the binomial theorem (614) can then be applied to (119)

$$\sin\, 2\,d_s \approx 2\, \sin d_s \left(1 - \frac{1}{2} \sin^2\, d_s\right) \tag{120}$$

yielding the sought after relationship.

9.2 Derivation of the leading solar and lunar tidal constituents

In the following, the leading solar and lunar tidal constituents are derived by expressing $\overline{\zeta}_1$ and $\overline{\zeta}_2$ by means a sum of single – and temporally varying – sine og cosine functions.

9.2.1 Expressing $\overline{\zeta}_1$ in terms of simple harmonics

With the above definitions and simplification, (72) can be expressed by means of ϕ_P , d_s , ϵ_s , λ_s , C_s and t, with the number in parenthesis above the equality signs showing to the expression/identity used:

$$\begin{aligned}\overline{\zeta}_1 &= \frac{1}{2}\sin(2\phi_P)\sin 2\,d_s\cos C_s \\ &\overset{(120)}{=} \sin(2\phi_P)\sin d_s\left(1-\frac{1}{2}\sin^2\,d_s\right)\cos C_s \\ &\overset{(113)}{=} \sin(2\phi_P)\sin\epsilon_s\left(\sin\lambda_s-\frac{1}{2}\sin^2\epsilon_s\sin^3\lambda_s\right)\cos C_s \\ &\overset{(632)}{=} \sin(2\phi_P)\sin\epsilon_s\left[\left(1-\frac{3}{8}\sin^2\epsilon_s\right)\sin\lambda_s+\frac{1}{8}\sin^2\epsilon_s\sin 3\lambda_s\right]\cos C_s \\ &\overset{(118)}{=} \sin(2\phi_P)\sin\epsilon_s\left[\left(1-\frac{3}{8}\sin^2\epsilon_s\right)\sin\omega_3 t+\frac{1}{8}\sin^2\epsilon_s\sin 3\omega_3 t\right]\cos(\omega_s-\omega_3)t \quad (121)\end{aligned}$$

For $\epsilon_s = 23.5°$, the 1/8-term is small (about 2 percent contribution), so (121) can be approximated as

$$\begin{aligned}\overline{\zeta}_1 &= \sin(2\phi_P)\sin\epsilon_s\left(1-\frac{3}{8}\sin^2\epsilon_s\right)\sin\omega_3 t\,\cos\omega_0 t \\ &= \overline{\zeta}_1'\,\sin\omega_3 t\,\cos\omega_0 t \quad (122)\end{aligned}$$

In the above expression, $\overline{\zeta}_1'$ includes the time-independent contribution to $\overline{\zeta}_1$. Expression (122) states that the temporal variation of $\overline{\zeta}_1$ is governed by the basic speed $\omega_0 = \omega_s - \omega_3 = 15$ deg/msh, modulated by the (slow) speed $\omega_3 = 0.0411$ deg/msh.

The identity

$$\sin a - \sin b \overset{(625)}{=} 2\sin\left(\frac{1}{2}(a-b)\right)\cos\left(\frac{1}{2}(a+b)\right) \quad (123)$$

can be used to split the trignometric product in (122) into two simple harmonics. With

$$\frac{1}{2}(a-b) = \omega_3 t \quad \text{and} \quad \frac{1}{2}(a+b) = \omega_0 t \quad (124)$$

we obtain $a = (\omega_0 + \omega_3)t$ and $b = (\omega_0 - \omega_3)t$. Thus,

$$\overline{\zeta}_1 = \frac{\overline{\zeta}_1'}{2}\left[\sin(\omega_0+\omega_3)t - \sin(\omega_0-\omega_3)t\right] \quad (125)$$

Expression (125) can alternatively be put in the form

$$\overline{\zeta}_1 = \frac{\overline{\zeta}_1'}{2}\left[\sin((\omega_s-\omega_3)+\omega_3)t - \sin((\omega_s-\omega_3)-\omega_3)t\right] \quad (126)$$

The Sun's diurnal contribution has therefore speeds $\omega_0 - \omega_3$ and $\omega_0 + \omega_3$ (from 125), symmetrically distributed about the basic speed $\omega_0 = \omega_s - \omega_3$ (expression 126).

In summary, $\overline{\zeta}_1$ gives rise to two tidal constituents with amplitude $\overline{\zeta}_1'/2$, originating from Earth's rotation around it's rotation axis and the Sun's half-yearly variation of the declination angle. The two constituents are called the solar declination constituents:

$\mathbf{K_1}$ Declinational diurnal constituent (will also get a contribution from the Moon, see Sec. 9.4); it's speed is $\omega_0 + \omega_3 = \sigma_s = 15.0411$ deg/msh and the period is 23.9344 msh.

$\mathbf{P_1}$ Principal (main) solar declinational diurnal constituent; it's speed is $\omega_0 - \omega_3 = \sigma_s - 2\,\sigma_3 = 14.9589$ deg/msh and period is 24.0659 msh.

9.2.2 Expressing $\overline{\zeta}_2$ in terms of simple harmonics

$\overline{\zeta}_2$ can rewritten as

$$\begin{aligned}
\overline{\zeta}_2 &= \frac{1}{2}\cos^2\phi_P\,\cos^2 d_s\,\cos 2C_s \\
&\overset{(618)}{=} \frac{1}{2}\cos^2\phi_P\,(1-\sin^2 d_s)\cos 2C_s \\
&\overset{(113)}{=} \frac{1}{2}\cos^2\phi_P\,(1-\sin^2\epsilon_s\sin^2\lambda_s)\cos 2C_s \\
&\overset{(631)}{=} \frac{1}{2}\cos^2\phi_P\left[1-\frac{1}{2}\sin^2\epsilon_s\,(1-\cos 2\lambda_s)\right]\cos 2C_s
\end{aligned} \tag{127}$$

If (127) is split in parts with and without λ_s-dependency, one obtains

$$\begin{aligned}
\overline{\zeta}_2 &= \frac{1}{2}\cos^2\phi_P\left[\left(1-\frac{1}{2}\sin^2\epsilon_s\right)\cos 2C_s + \frac{1}{2}\sin^2\epsilon_s\,\cos 2\lambda_s\,\cos 2C_s\right] \\
&= \frac{1}{2}\cos^2\phi_P\left[\left(1-\frac{1}{2}\sin^2\epsilon_s\right)\cos 2\omega_0 t\right. \\
&\qquad\qquad \left.+\frac{1}{2}\sin^2\epsilon_s\,\cos 2\,\omega_3 t\,\cos 2\omega_0 t\right]
\end{aligned} \tag{128}$$

For the last equality, the angular speeds from Sec. 9.1.1 have been introduced.

Similarly to $\overline{\zeta}_1$, the $\cos 2\,\omega_3 t\,\cos 2\omega_0 t$ factor in the last term of (128) consists of a temporal variation with basic speed $2\omega_0 = 2\,(\omega_s - \omega_3)$, slowly modulated by variation with speed $2\,\omega_3$. These two temporal variations can be split into two simple harmonics by means of the identity

$$\cos a + \cos b \overset{(626)}{=} 2\cos\left(\frac{1}{2}(a+b)\right)\cos\left(\frac{1}{2}(a-b)\right) \tag{129}$$

With the choice

$$\frac{1}{2}(a-b) = 2\omega_3 \quad \text{and} \quad \frac{1}{2}(a+b) = 2\omega_0 \tag{130}$$

one obtains $a = 2\,(\omega_0+\omega_3)$ and $b = 2\,(\omega_0-\omega_3)$. Consequently,

$$\begin{aligned}
\overline{\zeta}_2 &= \frac{1}{2}\cos^2\phi_P\left[\left(1-\frac{1}{2}\sin^2\epsilon_s\right)\cos 2\omega_0 t\right. \\
&\qquad +\frac{1}{4}\sin^2\epsilon_s\,\cos(2(\omega_0+\omega_3)t) \\
&\qquad \left.+\frac{1}{4}\sin^2\epsilon_s\,\cos(2(\omega_0-\omega_3)t)\right]
\end{aligned} \tag{131}$$

In summary, $\overline{\zeta}_2$ gives rise to three semi-diurnal tidal constituents, originating from Earth's rotation around it's rotation axis, modulated by the declination of the Sun:

$\mathbf{S_2}$ is the principal (main) solar semi-diurnal tide with speed $2\,\omega_0 = 2\,(\omega_s - \omega_3) = 30$ deg/msh and a period of 12 msh.

$\mathbf{K_2}$ is the semi-diurnal declination tide from the Sun and Moon (the latter contribution is given in Sec. 9.4), often called the lunisolar semi-diurnal constituent. It's peed is $2(\omega_0 + \omega_3) = 2\,\omega_s = 30.0822$ deg/msh and the period is 11.9672 msh.

The third constituent is the second semi-diurnal solar declination tide. It's speed is $2(\omega_0 - \omega_3) = 2\,(\omega_s - 2\,\omega_3) = 29.9178$ deg/msh and the period is 12.0330 msh.

Since $\epsilon_s = \text{const} = 23^\circ\,27'$, $\sin^2\epsilon_s = 0.16$, so the $2\,\omega_s$ and $2\,(\omega_s - 2\,\omega_3)$ constituents in (131) are small compared to $\mathbf{S_2}$. But since $\mathbf{K_2}$ gets a similar contribution from the Moon, this constituent is larger than the third constituent. One may therefore neglect the third component, but not $\mathbf{K_2}$.

9.3 Sun's leading tidal constituents

The following table gives an overview of the leading solar constituents.

Angular speed		**Period**	**Tidal constituent**	
Symbol	(deg/msh)	(msh)		
$\sigma_s - 2\,\sigma_3$	14.9589	24.0660	$\mathbf{P_1}$	Principal diurnal solar tide
σ_s	15.0411	23.9344	$\mathbf{K_1}$	Diurnal (solar+lunar) declination tide
$2\,\sigma_0 = 2\,(\sigma_s - \sigma_3)$	30.0000	12.0000	$\mathbf{S_2}$	Principal semi-diurnal solar tide
$2\,(\sigma_s - 2\,\sigma_3)$	27.8862	12.9096	(small)	Semi-diurnal solar declination tide
$2\,\sigma_s$	30.0822	11.9672	$\mathbf{K_2}$	Semi-diurnal (solar+lunar) declination tide

Table 3: Overview of the leading diurnal and semi-diurnal solar tidal constituents, excluding variations in the Earth's orbit around the Sun.

9.4 Moon's leading tidal constituents

Moon's declination varies from $18^\circ 18'$ to $28^\circ 36'$ in 18.6 years, but it can be considered constant for a fraction of 18.6 years, for instance for a half or a full year. In this case the Moon's tidal constituents follow those of the Sun derived above.

A full rotation of the Moon around the Earth as seen from a distant star gives the Moon's sidereal period $T_l = 27.3216$ msd (see Sec. 8.1). The Moon's angular speed is then

$$\frac{360^\circ}{27.3216\,\text{msd}} = 0.5490\,\frac{\text{deg}}{\text{msh}} \stackrel{\text{def}}{=} \sigma_2 \tag{132}$$

The relative hour angle follows that of the Sun by substituting σ_2 for σ_3 in (117)

$$\sigma_s - \sigma_2 = \sigma_1 = 14.4921\,\frac{\text{deg}}{\text{msh}} \tag{133}$$

where $\sigma_s = 15.0411$ deg/msd is the sidereal rotation speed of the Earth. Furthermore, the daily lunar period is 24 h 50 min, and the half-daily lunar period is 12 h 25 min (see 97).

Duplicating the results from the Sun and using

$$\lambda_l = \sigma_2\,t \quad \text{and} \quad C_l = \sigma_1\,t \tag{134}$$

one obtains the values as listed in Table 4.

Angular speed		Period	Tidal constituent	
Symbol	(deg/msh)	(msh)		
$\sigma_s - 2\,\sigma_2$	13.9431	25.8192	$\mathbf{O_1}$	Principal diurnal lunar tide
σ_s	15.0411	23.9344	$\mathbf{K_1}$	Diurnal (solar+lunar) declination tide
$2\,\sigma_1 = 2\,(\sigma_s - \sigma_2)$	28.9842	12.4206	$\mathbf{M_2}$	Principal semi-diurnal lunar tide
$2\,(\sigma_s - 2\,\sigma_2)$	27.8862	12.9096	(small)	Semi-diurnal lunar declination tide
$2\,\sigma_s$	30.0822	11.9672	$\mathbf{K_2}$	Semi-diurnal (solar+lunar) declination tide

Table 4: Overview of the major diurnal and semi-diurnal lunar tidal constituents, excluding variations in the Moon's orbit around the Earth.

9.5 Magnitude of the $\mathbf{O_1}$ and $\mathbf{K_1}$ constituents

The diurnal declination constituent $\mathbf{K_1}$ has both lunar and solar contributions, with the relative magnitude of the $\mathbf{O_1}$ and $\mathbf{K_1}$ given in table 7. **/Tab 4.1a/**

Table 5: Characteristics of the $\mathbf{O_1}$ and $\mathbf{K_1}$ tides. From Table 4.1a in Pugh & Woodworth (2014).

Tide	Period	Speed	Symbol	Relative magnitude	Name
	(msh)	(deg/msh)		($\mathbf{M_2}$ = 1.0000)	
$\mathbf{O_1}$	25.8194	13.9430	$\sigma_s - 2\,\sigma_2$	0.4151	Principal lunar
$\mathbf{K_1}$	23.9344	15.0411	σ_s	0.399	Lunar diurnal declination
$\mathbf{K_1}$	23.9344	15.0411	σ_s	0.1852	Solar diurnal declination
				Sum = 1.000	

It follows that the magnitude of $\mathbf{O_1}$ and $\mathbf{K_1}$ equals the magnitude of $\mathbf{M_2}$, illustrating the important role of $\mathbf{O_1}$ and $\mathbf{K_1}$ where diurnal tidal variations are present.

9.6 The lunar semi-monthly tide $\mathbf{M_f}$

The slowly varying declination part of the equilibrium tide, from expression (71),

$$\bar{\zeta}_0 = \frac{3}{2}\left(\sin^2\phi_P - \frac{1}{3}\right)\left(\sin^2 d_l - \frac{1}{3}\right) \tag{135}$$

has a temporal variation with speed of $2\,\sigma_2$. This follows directly from

$$\sin^2 d_l \overset{(113)}{=} \sin^2\epsilon_l\,\sin^2\lambda_l \overset{(623)}{=} \frac{1}{2}\sin^2\epsilon_l\,(1-\cos(2\,\lambda_l)) \overset{(134)}{=} \frac{1}{2}\sin^2\epsilon_l\,(1-\cos(2\,\sigma_2\,t)) \tag{136}$$

The resulting constituent – the lunar semi-monthly tide $\mathbf{M_f}$ – has a period

$$T = \frac{360^\circ}{2\,\sigma_2} = 13.66 \text{ msd} \tag{137}$$

The magnitude of $\mathbf{M_f}$ relative to that of $\mathbf{M_2}$ is 0.1723 (Table 4.1 in Pugh & Woodworth, 2014). One can therefore expect the declination tide $\mathbf{M_f}$ to be present in any time series of sea surface height.

Furthermore, the speed of $\mathbf{M_f}$ is identical to the speed of the interacting diurnal constituents $\mathbf{O_1}$ and $\mathbf{K_1}$, see Sec. 10.2. Extraction of $\mathbf{M_f}$ therefore requires removal of the $\mathbf{O_1}$ and $\mathbf{K_1}$ harmonics, and thereby the interaction between the two, from a given sea surface height time series.

Regarding the name $\mathbf{M_f}$: $\mathbf{M}$ denotes the Moon and the subscript $\mathbf{f}$ hints to the (near) fortnightly period 13.66 msd.

9.7 Doodson's system for labelling tidal constituents

To facilitate a tabular overview of the (many) tidal constituents, Doodson invented a 6-digit system characterising the speed of the constituents. The speed of the individual contributions σ_n can be described by means of 6 basic frequencies $\sigma_1 ... \sigma_6$, **/p. 4.1/**

$$\boxed{\sigma_n = i_a \sigma_1 + i_b \sigma_2 + ... + i_f \sigma_6} \tag{138}$$

where $i_{a...f}$ are integers.

The frequencies $\sigma_1 ... \sigma_6$ in the above expression are as listed in table 2 (and are shown below), with the two alternative expressions for σ_1 from (101):

$\sigma_1 = \sigma_s - \sigma_2 = \sigma_0 - \sigma_2 + \sigma_3$	$\sigma_1 = 14.4921$ deg/msh, $\sigma_s = 15.0411$ deg/msh
σ_2	$\sigma_2 =$ 0.5490 deg/msh
σ_3	$\sigma_3 =$ 0.0411 deg/msh
$\sigma_4 ... \sigma_6$	Frequencies related to the geometry of the slow planetary motions (not included here)

Table 6 summarises the Doodson coefficients for four of the leading tidal constituents. As an example, the speed of the S_2 tide, $2(\sigma_s - \sigma_3)$, can be expressed in terms of $\sigma_{1,...,6}$:

$$2\,(\sigma_s - \sigma_3) = 2\,(\sigma_1 + \sigma_2 - \sigma_3) = 2\,\sigma_1 + 2\,\sigma_2 - 2\,\sigma_3 + 0\,\sigma_4 + 0\,\sigma_5 + 0\,\sigma_6 \tag{139}$$

yielding the Doodson coefficients $i_a = 2, i_b = 2, i_c = -2, i_{d,e,f} = 0$, or 22-2.000, where the period mark in the number sequence is included to facilitate reading.

To circumvent negative coefficients, it is common practice to add 5 to the coefficients $i_b ... i_f$, see the right-hand column in Table 6.

Table 6: Overview of the major solar and lunar tides expressed in terms of the Doodson coefficients.

Tide	**Period** (msh)	**Speed** (deg/msh)	**Symbol**	**Doodson coefficients** $i_a i_b i_c . i_d i_e i_f$	**Non-negative coefficients** (add 5 to $i_b ... i_f$)
$\mathbf{M_2}$	12.4206	28.9841	$2\,(\sigma_s - \sigma_2)$	200.000	255.555
$\mathbf{S_2}$	12.0000	30.0000	$2\,(\sigma_s - \sigma_3)$	22-2.000	273.555
$\mathbf{K_1}$	23.9344	15.0411	σ_s	110.000	165.555
$\mathbf{O_1}$	25.8194	13.9430	$\sigma_s - 2\,\sigma_2$	1-10.000	145.555

10 Two fortnightly tidal signals

In regions where the $\mathbf{M_2}$ and $\mathbf{S_2}$ tides dominate, the combined $\mathbf{M_2}+\mathbf{S_2}$ tide generates a prominent fortnightly signal with period 14.77 msd known as the *spring tide*[10] when the sea surface hight is particularly high and the *neap tide*[11] when the surface is particularly low, see Fig. 17.

The spring tide occurs when the three-body Earth-Moon-Sun system is closely aligned along a line – often referred to *syzygy*[12] – implying new or full Moon seen from the Earth. The neap tide occurs when the Moon and the Sun are approximately at right angles to each other seen from the Earth (at first-quarter and third-quarter Moon).

For regions dominated by diurnal variations and similarly to the spring/neap tide, the lunar declination terms $\mathbf{O_1}$ and $\mathbf{K_1}$ tides interact, resulting in sea surface height variations with a period of 13.66 msd. This period is identical – and thus indistinguishable from a harmonic analysis point of view – to the $\mathbf{M_f}$ tide (see Sec. 9.6).

10.1 The spring/neap tide

The sea surface elevation $Z(t)$ governed by the combined $\mathbf{M_2}+\mathbf{S_2}$ tide can be expressed as

$$Z(t) = Z_0 + H_{\mathrm{M}_2}\cos(2\,\sigma_1 t - g_{\mathrm{M}_2}) + H_{\mathrm{S}_2}\cos(2\,\sigma_0 t - g_{\mathrm{S}_2}) \tag{140}$$

Here Z_0 is the sea surface elevation in the absence of tides, and H_{M_2}, $2\,\sigma_1$ and g_{M_2} are the amplitude, frequency and (constant) phase of the $\mathbf{M_2}$ tide. Similarly, H_{S_2}, $2\,\sigma_0$ and g_{S_2} are the corresponding quantities for the $\mathbf{S_2}$ tide.

The $\mathbf{S_2}$ argument can always be expressed similarly to the leading $\mathbf{M_2}$ argument $2\,\sigma_1 t - g_{\mathrm{M}_2}$:

$$Z(t) = Z_0 + H_{\mathrm{M}_2}\cos(2\,\sigma_1 t - g_{\mathrm{M}_2}) + H_{\mathrm{S}_2}\cos(2\,\sigma_1 t - g_{\mathrm{M}_2} - \theta) \tag{141}$$

Consequently, by comparing (140) and (141), **/p. 75/**

$$\boxed{2\,\sigma_0 t - g_{\mathrm{S}_2} = 2\,\sigma_1 t - g_{\mathrm{M}_2} - \theta} \tag{142}$$

or

$$\theta(t) = 2(\sigma_1 - \sigma_0)t + g_{\mathrm{S}_2} - g_{\mathrm{M}_2} \tag{143}$$

As will be shown in expression (145) below, *the temporally varying part of* θ*, the angular frequency* $2(\sigma_1 - \sigma_0)$*, gives rise to the the fortnightly period, or to the spring/neap tidal signal.* The difference $\sigma_0 - \sigma_1 = 0.5079$ deg/msh (values from table 2), yields the spring/neap period

$$T = 14.7667 \text{ msd} \tag{144}$$

Note that the period of the spring/neap tide is not a basic tidal constituent in itself, but occurs as an interference between the two tidal constituents $\mathbf{M_2}$ and $\mathbf{S_2}$ with closely matching periods and comparable amplitudes. The combined $\mathbf{M_2}+\mathbf{S_2}$ tide is prominent in many places, and is thus an important part of the total tidal variation.

[10] *springflo, springfjære*
[11] *nippflo, nippfjære*
[12] *syzygy* på norsk, også...

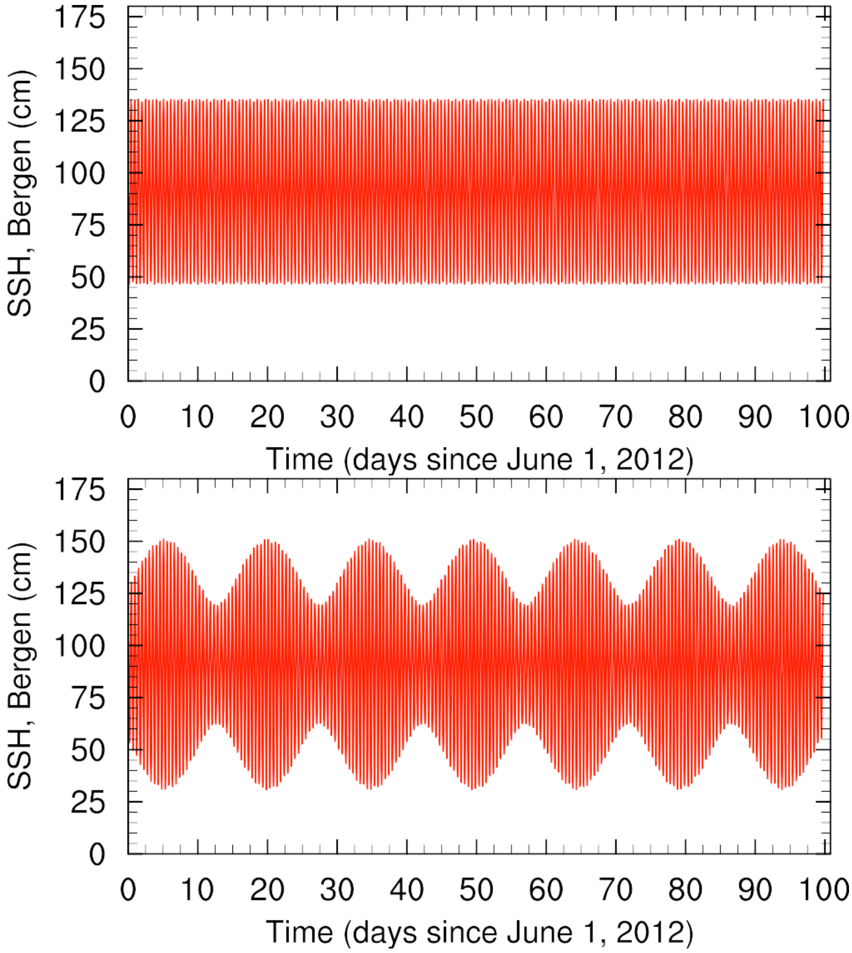

Figure 17: Computed $\mathbf{M_2}$ (top panel) and the combined $\mathbf{M_2 + S_2}$ tides for Bergen, Norway, based on harmonic analysis of observed sea level acquired from `https://www.kartverket.no/sehavniva/`.

10.1.1 Derivation of the spring/neap tide

Expansion of the second cosine-term in (141) gives

$$\begin{aligned} Z(t) &\stackrel{(620)}{=} Z_0 + H_{M_2}\cos(2\,\sigma_1 t - g_{M_2}) \\ &\quad + H_{S_2}\{\cos(2\,\sigma_1 t - g_{M_2})\cos\theta + \sin(2\,\sigma_1 t - g_{M_2})\sin\theta\} \\ &= Z_0 + (H_{M_2} + H_{S_2}\cos\theta)\cos(2\,\sigma_1 t - g_{M_2}) + H_{S_2}\sin\theta\,\sin(2\,\sigma_1 t - g_{M_2}) \end{aligned} \tag{145}$$

The above expression consists of slow amplitude modulations given by the factors $H_{M_2}+H_{S_2}\cos\theta$ and $H_{S_2}\sin\theta$ – both with period 14.7667 msd, originating from the speed difference $2(\sigma_1-\sigma_0)$ – and semi-diurnal variations given by the argument $2\,\sigma_1 t - g_{M_2}$.

The slow amplitude modulation can conveniently be expressed by means of a time dependent amplitude α' and argument β' :

$$H_{M_2} + H_{S_2}\cos\theta = \alpha'\cos\beta' \tag{146}$$

$$H_{S_2}\sin\theta = \alpha'\sin\beta' \tag{147}$$

β' can be determined upon dividing (147) by (146), yielding

$$\tan\beta' = \frac{H_{S_2}\sin\theta}{H_{M_2}+H_{S_2}\cos\theta}\,, \qquad \text{so} \qquad \beta' = \arctan\left(\frac{H_{S_2}\sin\theta}{H_{M_2}+H_{S_2}\cos\theta}\right) \tag{148}$$

Likewise, α' can be determined by squaring and adding (146) and (147):

$$\alpha' = \sqrt{H_{M_2}^2 + H_{S_2}^2 + 2H_{M_2}H_{S_2}\cos\theta)} \tag{149}$$

$Z(t)$ can now be cast in the compact form (by means of the identity 620): **/4.8/**

$$\boxed{Z(t) = Z_0 + \alpha'\cos(2\,\sigma_1 t - g_{M_2} - \beta')} \tag{150}$$

10.1.2 Amplitude and range

From expression (149), the largest amplitude of the $\mathbf{M_2}+\mathbf{S_2}$ tide occurs when $\cos\theta = 1$:

$$\alpha'_{\text{max}} = H_{M_2} + H_{S_2} \tag{151}$$

In this case, the (spring) tidal range varies between $Z_0 - (H_{M_2} + H_{S_2})$ and $Z_0 + (H_{M_2} + H_{S_2})$.

Similarly, the smallest amplitude occurs when $\cos\theta = -1$:

$$\alpha'_{\text{min}} = H_{M_2} - H_{S_2} \tag{152}$$

Thus, the neap tidal range varies between $Z_0 - (H_{M_2} - H_{S_2})$ and $Z_0 + (H_{M_2} - H_{S_2})$.

With $H_{M_2} \approx 2.2\,H_{S_2}$ (see Sec. 6.1), this implies that the spring and neap tides have amplitudes $\alpha'_{\text{max}} \approx 3.2\,H_{S_2}$ and $\alpha'_{\text{min}} \approx 1.2\,H_{S_2}$, respectively, with a ratio

$$\frac{\alpha'_{\text{max}}}{\alpha'_{\text{min}}} \approx 2.5 \tag{153}$$

10.1.3 Timing of maximum amplitude

The largest amplitude α'_{max} of the spring tide occurs when $\theta = 0$ or any 2π multiples of θ (see 149). From expression (143), maximum amplitude in the case of $\theta = 0$ occurs at **/4.9/**

$$\boxed{t_{\mathrm{max}} = \frac{g_{\mathrm{S}_2} - g_{\mathrm{M}_2}}{2(\sigma_0 - \sigma_1)}} \tag{154}$$

10.1.4 Age of the tide

The arguments of the $\mathbf{M_2}$ tide from (140) and the combined $\mathbf{M_2} + \mathbf{S_2}$ tide from (150) are, respectively,

$$2\,\sigma_1 t - g_{\mathrm{M}_2} \qquad \text{and} \qquad 2\,\sigma_1 t - g_{\mathrm{M}_2} - \beta' \tag{155}$$

A time-varying time lag $\Delta = \Delta(t)$ between the $\mathbf{M_2}$ and $\mathbf{M_2} + \mathbf{S_2}$ tides can thus be determined from the expression

$$2\sigma_1(t + \Delta) - g_{\mathrm{M}_2} = 2\,\sigma_1 t - g_{\mathrm{M}_2} - \beta' \tag{156}$$

resulting in

$$\Delta(t) = -\frac{\beta'(t)}{2\,\sigma_1} \tag{157}$$

The time lag between the two tides is largest when β' reaches its maximum or minimum value, i.e., when

$$\frac{d\beta'}{dt} = 0 \tag{158}$$

with β' defined by (148).

With the substitution

$$x(t) = \frac{H_{\mathrm{S}_2} \sin\theta(t)}{H_{\mathrm{M}_2} + H_{\mathrm{S}_2}\cos\theta(t)} \tag{159}$$

expression (148) becomes

$$\beta' = \arctan x \tag{160}$$

Using the derivative of arctangent (expression 643), we obtain

$$\frac{d\beta'}{dt} = \frac{d\beta'}{dx}\frac{dx}{dt} = \frac{1}{1+x^2}\,\frac{dx}{dt} = 0 \tag{161}$$

Maximising β', for a general θ, is therefore given by $dx/dt = 0$. Vanishing derivative of (159) with respect to t leads to the relationship

$$\cos\theta = -\frac{H_{\mathrm{S}_2}}{H_{\mathrm{M}_2}} \tag{162}$$

From the equilibrium theory, see Sec. 6.1, $H_{\mathrm{M}_2} = 0.268$ m and $H_{\mathrm{S}_2} = 0.123$ m. Thus,

$$\cos\theta = -0.46\,, \qquad \text{or} \qquad \theta = \pm 2.05 \text{ rad} \tag{163}$$

H_{M_2}, H_{S_2} and θ inserted into (148) gives

$$\tan\beta' \approx \pm 0.52 \qquad \text{or} \qquad \beta' \approx \pm 48 \text{ rad (or} \pm 27 \text{ deg)} \tag{164}$$

Largest time lag is obtained for $\beta' = \pm 27$ deg, yielding maximum time lag from (157) of /p. 76/

$$\boxed{\Delta_{\max} = \frac{\mp 27 \text{ deg}}{28.9842 \text{ deg/msh}} = \mp 0.94 \text{ msh} = \mp 57 \text{ min}} \tag{165}$$

Thus, the combined $\mathbf{M_2} + \mathbf{S_2}$ tide lags/leads the $\mathbf{M_2}$ tide with slightly less than 1 hour at most. This time difference – the maximum lag/lead between the new and the full Moon and the maximum spring tidal range – is called the *age of the tide*[13].

The time lag between the $\mathbf{M_2}$ and $\mathbf{M_2} + \mathbf{S_2}$ tides, derived from expression (157), is plotted in Fig. 18. During the lunar synodic period $T_{\text{synodic}} = 29.5307$ msd (Sec. 8.1), the time lag maxima occurs at the fraction $\pm 2.05 \text{ rad}/(2\,\pi)$ of the period T_{synodic}, or at

$$\frac{2.05 \text{ rad}}{2\,\pi} T_{\text{synodic}} \approx 9.6 \text{ msd} \qquad \text{and} \qquad \frac{2\,\pi - 2.05 \text{ rad}}{2\,\pi} T_{\text{synodic}} \approx 19.9 \text{ msd} \tag{166}$$

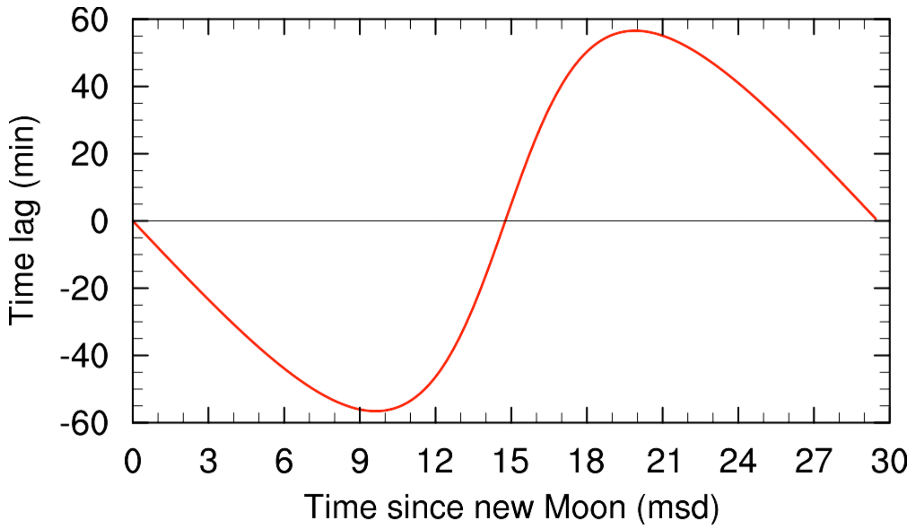

Figure 18: Illustration of the age of the tide Δ, from expression (157), through a full lunar synodic period of 29.5307 msd. Maximum time lag of slightly less than 1 hr, called the age of the tide, occurs about 9.6 and 19.9 msd after full Moon.

Based on observed tidal variations, the age of the tide can be much longer than approximately 1 hr, in many places several days. The difference between the theory presented here and the observed variations are to a large degree caused by the speed of the tidal signal governed by gravity waves. The latter has a phase speed of $\sqrt{g\,D}$ (where g is gravity and D is the water depth; see Chap. 20) which, generally, are much slower than the sub-lunar point on Earth.

10.1.5 A note

For any two waves with closely matching speed and amplitude, the beating frequency equals the speed difference between the two waves. This follows from the derivation of the group speed as outlined in appendix F.5, where the sum of two comparable waves (see expression 602) leads to a 'modified' wave with a slowly changing amplitude and speed (expression 609).

[13] *tidevannets alder*

10.2 Interaction of the diurnal $\mathbf{O_1}$ and $\mathbf{K_1}$ tides

Similar to the procedure for the fortnightly spring/neap tide discussed above, the speed of the interacting $\mathbf{O_1}$ and $\mathbf{K_1}$ tides is based on the speed difference between the two (see expressions 143 and 145): /p. 69/

$$\sigma_s - (\sigma_s - 2\,\sigma_2) = 2\,\sigma_2 \tag{167}$$

or twice the lunar sidereal month. The corresponding period is (values in table 2)

$$T = \frac{360°}{2\,\sigma_2} = 13.66 \text{ msd} \tag{168}$$

The resulting period is thus identical to that of the lunar semi-monthly tide $\mathbf{M_f}$ disussed in Sec. 9.6.

10.3 Shallow water terms

In shallow waters, non-linear interactions of the basic tidal constituents lead to so-called shallow water terms. These terms are proportional to the square (and cube, and higher order) of the amplitude of, e.g., the sum of the $\mathbf{M_2}$ and $\mathbf{S_2}$ constituents: /4.2.3/

$$\kappa_2\,[H_{\mathrm{M}_2}\,\cos(2\,\omega_1 t) + H_{\mathrm{S}_2}\,\cos(2\,\omega_0 t)]^2 \tag{169}$$

Here κ_2 is a small constant scaling the square-law interaction and, in our case, having a given (known) value.

The above equation becomes

$$\kappa_2\,[H^2_{\mathrm{M}_2}\,\cos^2(2\,\omega_1 t) + 2\,H_{\mathrm{M}_2}\,H_{\mathrm{S}_2}\,\cos(2\,\omega_1 t)\,\cos(2\,\omega_0 t) + H^2_{\mathrm{S}_2}\,\cos^2(2\,\omega_0 t)] \tag{170}$$

By means of identity (624),

$$H^2_{\mathrm{M}_2}\,\cos^2(2\,\omega_1 t) = \frac{1}{2}\,H^2_{\mathrm{M}_2}\,(1 + \cos(4\,\omega_1 t)) \tag{171}$$

and

$$H^2_{\mathrm{S}_2}\,\cos^2(2\,\omega_0 t) = \frac{1}{2}\,H^2_{\mathrm{S}_2}\,(1 + \cos(4\,\omega_0 t)) \tag{172}$$

The $\cos(2\omega_1 t)\,\cos(2\omega_0 t)$-term can be expressed in terms of simple harmonics by adding the two identities (620):

$$\cos(a+b) + \cos(a-b) = 2\,\cos a \cos b \tag{173}$$

Put together, this gives

$$\kappa_2 \begin{bmatrix} \frac{1}{2}H^2_{\mathrm{M}_2} + \frac{1}{2}H^2_{\mathrm{S}_2} + \frac{1}{2}H^2_{\mathrm{S}_2}\,\cos(4\,\omega_1 t) + \frac{1}{2}H^2_{\mathrm{S}_2}\,\cos(4\,\omega_0 t) \\ +\,H^2_{\mathrm{M}_2}\,H^2_{\mathrm{S}_2}\,\cos(2\,(\omega_1+\omega_0)t) + H^2_{\mathrm{M}_2}H^2_{\mathrm{S}_2}\,\cos(2\,(\omega_0-\omega_1)t) \end{bmatrix} \tag{174}$$

The harmonics with speed $4\,\omega_1$ and $4\,\omega_0$ are named $\mathbf{M_4}$ and $\mathbf{S_4}$, respectively, with the subscript 4 denoting the speed factor relative to the basic speeds $\omega_{0,1}$.

Furthermore, the harmonics with speed $2\,(\omega_1+\omega_0)$ is called $\mathbf{MS_4}$, whereas that with speed $2\,(\omega_0-\omega_1)$ is called $\mathbf{MS_f}$, where the latter subscript indicates the fortnightly period.

10.4 Non-gravitational harmonics

Analysis of time series typically shows periodicities that can not be explained by gravitational contributions, or contributions that add to the basic tidal signals. Some of the most prominent of these signals are:

Solar diurnal signal $\mathbf{S_1}$ Period is 24.0000 msh. This is a radiational harmonics, caused by solar heating/cooling, and/or diurnal variations in wind strength/direction, and/or diurnal variations in sea level pressure. **/Sec. 5.5/**

Solar annual signal $\mathbf{S_a}$ Period is 365.2422 msd. Main contribution is radiational (seasonal solar heating/cooling), with a smaller gravitational contribution. **/Sec. 4.3.3/**

Solar semi-annual signal $\mathbf{S_{sa}}$ Period is 182.6213 msd. Main contribution is radiational (seasonal solar heating/cooling), with a smaller gravitational contribution. **/Sec. 4.3.3/**

10.5 Summary of the mentioned non-leading harmonics

Characteristics of the fortnightly constituent $\mathbf{M_f}$ (Sec. 9.4), the smaller and larger elliptical lunar (Sec. E.1), and the shallow water constituents (Sec. 10.3) are listed below. **/Tab. 4.4/**

Table 7: Characteristics of some of the non-leading harmonics. Based on Table 4.4 in Pugh & Woodworth (2014).

	Origin	**Speed**		**Period**	**Name**
			(deg/msh)	360°/speed	
$\mathbf{N_2}$	$\overline{R}_l/R_l$	$2\,\sigma_1 - \sigma_2 + \sigma_4$	28.4397	12.6584 msd	Larger elliptical lunar
$\mathbf{L_2}$	$\overline{R}_l/R_l$	$2\,\sigma_1 + \sigma_2 - \sigma_4$	29.5285	12.1916 msd	Smaller elliptical lunar
$\mathbf{S_1}$	Solar heating	σ_0	15.0000	24.000 msd	Solar diurnal, radiational
$\mathbf{S_a}$	Solar heating	σ_3	0.0411	365.2422 msd	Solar annual, radiational
$\mathbf{S_{sa}}$	Solar heating	$2\,\sigma_3$	0.0822	182.6213 msd	Solar semi-annual, radiational
$\mathbf{M_f}$	$\overline{\zeta}_0$	$2\,\sigma_2$	1.0980	13.6612 msd	Lunisolar fortnightly
$\mathbf{MS_f}$	$\mathbf{M_2}$, $\mathbf{S_2}$	$2\,(\sigma_0 - \sigma_1)$	1.0158	14.7667 msd	Lunisolar synodic fortnightly
$\mathbf{M_4}$	$\mathbf{M_2}$	$4\,\sigma_1$	57.9682	6.2103 msh	Shallow water overtide, lunar
$\mathbf{S_4}$	$\mathbf{S_2}$	$4\,\sigma_0$	60.0000	6.0000 msh	Shallow water overtide, solar
$\mathbf{MS_4}$	$\mathbf{M_2} + \mathbf{S_2}$	$2\,(\sigma_1 + \sigma_0)$	58.9841	6.1033 msh	Shallow water quarter diurnal

11 Tidal forcing in the momentum equation

The full 3-dimensional tidal flow, taking into account bathymetry and coastlines, can be described by the primitive equations. The 3-dimensional momentum equation can be expressed in the form (M & P (2008), Chap. 6)

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + f\hat{\mathbf{z}} \times \mathbf{u} = -g\,\hat{\mathbf{z}} + \mathcal{F} \tag{175}$$

In (175), $\mathbf{u}$ is the 3-dimensional velocity field, $D\mathbf{u}/Dt$ is the total derivative of $\mathbf{u}$, p is pressure, ρ is density, f is the Coriolis parameter, $\hat{\mathbf{z}}$ is the radial, outward-directed unit vector on a sphere, and $\mathcal{F}$ is friction.

The tidal forcing from the Moon and the Sun at the surface of the Earth can be added to (175) by introducing the pressure force (40) caused by the tidal elevation $\overline{\zeta}$ (54):

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + f\hat{\mathbf{z}} \times \mathbf{u} = -g\,\hat{\mathbf{z}} - g\nabla\overline{\zeta} + \mathcal{F} \tag{176}$$

where

$$\overline{\zeta} = \frac{3}{2}\frac{r^4}{m_e}\left[\frac{m_l}{R_l^3}\left(\cos^2\phi_l - \frac{1}{3}\right) + \frac{m_s}{R_s^3}\left(\cos^2\phi_s - \frac{1}{3}\right)\right] \tag{177}$$

In the expression for $\overline{\zeta}$, $\cos^2\phi - 1/3$ is written in terms of the latitude ϕ_P and the hour angle C_P of a position P on Earth's surface, as well as the declination angle d of the celestial body (see Sec. 7 and expression 69):

$$\cos^2\phi - \frac{1}{3} = \cos^2\phi_P\cos^2 d\cos^2 C_P + \frac{1}{2}\sin(2\,\phi_P)\sin(2\,d)\cos C_P + \sin^2\phi_P\sin^2 d - \frac{1}{3} \tag{178}$$

The hour ("longitude") angle C_P needs to be referenced to a fixed geographic point. A common reference point is the point of the vernal equinox[14] ♈ (also known as the first point of Aries), see Fig. 16.

The angles ϕ_P, C_s and δ can be computed in Fortran, C and Python, for any time, from e.g. the Naval Observatory Vector Astrometry Software (NOVAS, see Kaplan *et al.*, 2011, `http://aa.usno.navy.mil/software/novas/novas_info.php`).

The momentum equation (176), with expressions (177) and (178), together with the continuity equation, models the full 3-dimensional dynamics of the tide. The set of equations, incorporating realistic bathymetry, can only be solved numerically.

See also page 308 in Cushman-Roisin and Beckers, *Introduction to Geophysical Fluid Dynamics*, Academic Press, Elsevier, Amsterdam, 2011.

[14] *vårjevndøgn*

12 Laplace's Tidal Equations

When tidal forcing is introduced to the (quasi-)linearised version of the shallow water equations, the obtained equations are known as *Laplace's Tidal Equations* (LTE). Tidal flow is then described as the flow of a barotropic fluid, forced by the tidal pull from the Moon and the Sun. The phrase "shallow water equations" reflects that the wavelength of the resulting motion is large compared to the thickness of the fluid.

The horizontal components of the momentum equation and the continuity equation can then be expressed as

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial}{\partial x}(\zeta + \overline{\zeta}) \tag{179}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial}{\partial y}(\zeta + \overline{\zeta}) \tag{180}$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) = 0 \tag{181}$$

In the above equations, $-g\nabla\overline{\zeta}$ is the tidal forcing with $\overline{\zeta}$ obtained from (177) and (178), ζ is the surface elevation, and h is the ocean depth.

The horizontal momentum equations are linear, but inclusion of a friction term will typically turn the equations non-linear. Likewise, the divergence terms in the continuity equation are nonlinear because of the product uh and vh. Solution of LTE requires discretisation and subsequent numerical solution.

13 The tidal force expressed in terms of the tidal potential

The direct method described in Sec. 5 can be conveniently carried out by introducing the *tidal potential* as described below.

13.1 The potential of a conservative force

The gravitational force is an example of a *conservative force*, i.e., a force with the property that the work done in moving a particle from position a to position b is independent of the path taken (think of changes in the potential energy when an object in Earth's gravitational field is moved from one position to a new position; the path taken is irrelevant, only the height difference between the start and end positions matter).

It then follows that the gravitational force $\mathbf{F}$ can be expressed as the negative gradient of a potential (a scalar) Ω

$$\mathbf{F} = -\nabla\Omega \tag{182}$$

Once Ω is found, the resulting force can be readily obtained by taking the gradient of Ω. This simplification is a main motivation of introducing the potential, and it is commonly used in problems involving gravitation and electromagnetism.

13.2 Defining the tidal potential

From (18), we have that the tidal acceleration can be expressed as

$$\mathbf{a} = G\,m_l \left(\frac{\mathbf{q}}{q^3} - \frac{\mathbf{R}}{R^3} \right) \tag{183}$$

Let F_t denote the tidal force per unit mass (or acceleration). It follows then from the above expression that

$$\mathbf{F_t} = G\,m_l \left(\frac{\mathbf{q}}{q^3} - \frac{\mathbf{R}}{R^3} \right) \tag{184}$$

The task is therefore to determine the scalar function Ω, the tidal potential, satisfying

$$\mathbf{F_t} = -\nabla\Omega \tag{185}$$

13.2.1 Gradient of the factor $\mathbf{q}/q^3$

We start by searching for a scalar with the property that the gradient of the scalar equals $\mathbf{q}/q^3$. It follows from Fig. (4) that

$$\mathbf{q} = \mathbf{R} - \mathbf{r} \tag{186}$$

In the above expression, $\mathbf{R}$ is a constant (fixed) vector whereas $\mathbf{r}$ varies spatially as it can oriented at any location on Earth's surface.

In a Cartesian coordinate system with time-invariant unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$, we may express

$$\mathbf{r} = x\,\mathbf{e}_x + y\,\mathbf{e}_y + z\,\mathbf{e}_z \tag{187}$$

and

$$\mathbf{R} = R_x\,\mathbf{e}_x + R_y\,\mathbf{e}_y + R_z\,\mathbf{e}_z \tag{188}$$

Therefore,

$$\mathbf{q} = (R_x - x)\,\mathbf{e}_x + (R_y - y)\,\mathbf{e}_y + (R_z - z)\,\mathbf{e}_z \tag{189}$$

The magnitude of q becomes

$$q = |\mathbf{q}| = |\mathbf{R} - \mathbf{r}| = \left[(R_x - x)^2 + (R_y - y)^2 + (R_z - z)^2\right]^{1/2} \tag{190}$$

so

$$q^3 = \left[(R_x - x)^2 + (R_y - y)^2 + (R_z - z)^2\right]^{3/2} \tag{191}$$

Furthermore,

$$\nabla\left(\frac{1}{q}\right) = \frac{\partial}{\partial x}\left(\frac{1}{q}\right)\mathbf{e}_x + \frac{\partial}{\partial y}\left(\frac{1}{q}\right)\mathbf{e}_y + \frac{\partial}{\partial z}\left(\frac{1}{q}\right)\mathbf{e}_z \tag{192}$$

In the above expression

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{1}{q}\right) &= \frac{\partial}{\partial x}\left(\frac{1}{[(R_x - x)^2 + (R_y - y)^2 + (R_z - z)^2]^{1/2}}\right) \\ &= -\frac{1}{2}\,\frac{-2(R_x - x)}{[(R_x - x)^2 + (R_y - y)^2 + (R_z - z)^2]^{3/2}}\end{aligned} \tag{193}$$

Similar results are obtained for the y- and z-derivatives in (192). Consequently,

$$\nabla\left(\frac{1}{q}\right) = \frac{(R_x - x)\,\mathbf{e}_x + (R_y - y)\,\mathbf{e}_y + (R_z - z)\,\mathbf{e}_z}{q^3} = \frac{\mathbf{q}}{q^3} \tag{194}$$

The $\mathbf{q}/q^3$-term in (184) can therefore be expressed as $\nabla(1/q)$. This is a classical result from both electrodynamics and analysis involving gravity.

13.2.2 Gradient of the factor $\mathbf{R}/R^3$

The second term on the right-hand side of (184) can also be turned into the gradient of a scalar. For this we use that

$$\mathbf{r}\cdot\mathbf{R} = x\,R_x + y\,R_y + z\,R_z \tag{195}$$

Consequently,

$$\begin{aligned}\nabla(\mathbf{r}\cdot\mathbf{R}) &= \frac{\partial}{\partial x}(\mathbf{r}\cdot\mathbf{R})\,\mathbf{e}_x + \frac{\partial}{\partial y}(\mathbf{r}\cdot\mathbf{R})\,\mathbf{e}_y + \frac{\partial}{\partial z}(\mathbf{r}\cdot\mathbf{R})\,\mathbf{e}_x \\ &= R_x\,\mathbf{e}_x + R_y\,\mathbf{e}_y + R_z\,\mathbf{e}_z \\ &= \mathbf{R}\end{aligned} \tag{196}$$

The $\mathbf{R}/R^3$-term in (184) is therefore equivalent to $\nabla(\mathbf{r}\cdot\mathbf{R})$.

13.2.3 Resulting tidal potential

By inserting (194) and (196) into (184), we obtain

$$\mathbf{F_t} = -Gm_l\nabla\left(-\frac{1}{q} + \frac{\mathbf{r}\cdot\mathbf{R}}{R^3} + C\right) \tag{197}$$

where C is a constant.

C can be determined by noting that F_t vanishes at Earth's centre, see the discussion in Sec. 5.7. When $\mathbf{r} = 0$, $\mathbf{q}$ equals $\mathbf{R}$, and we get from (197) that $C = 1/R$. Therefore

$$\mathbf{F_t} = -Gm_l \nabla \left(-\frac{1}{q} + \frac{\mathbf{r} \cdot \mathbf{R}}{R^3} + \frac{1}{R} \right) = -\nabla \Omega \tag{198}$$

where the tidal potential Ω was introduced in (185). The tidal potential can therefore be put in the form

$$\Omega = Gm_l \left(-\frac{1}{q} + \frac{\mathbf{r} \cdot \mathbf{R}}{R^3} + \frac{1}{R} \right) \tag{199}$$

13.3 Tidal potential expressed in terms of the zenith angle

Next step is to express (199) by means of the zenith angle ϕ (see Fig. 4). For the $1/q$-term the procedure follows that of Sec. 5.8, starting with the law of cosines (19),

$$q^2 = R^2 + r^2 - 2\,R\,r\cos\phi = R^2 \left[1 + \frac{r}{R}\left(\frac{r}{R} - 2\,\cos\phi\right)\right] \tag{200}$$

Consequently,

$$q = R\left[1 + \frac{r}{R}\left(\frac{r}{R} - 2\,\cos\phi\right)\right]^{1/2} \tag{201}$$

The factor r/R has been introduced in the above expressions since it is a small number (appendix A), allowing for series expansion. The quantity $1/q$ can then be expended by applying the binomial theorem (614). We neglect terms proportional to and smaller than $(r/R)^3$, giving

$$\begin{aligned}
\frac{1}{q} &= \frac{1}{R}\left[1 + \frac{r}{R}\left(\frac{r}{R} - 2\,\cos\phi\right)\right]^{-1/2} \\
&= \frac{1}{R}\left[1 - \frac{1}{2}\frac{r}{R}\left(\frac{r}{R} - 2\,\cos\phi\right) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}\left(\frac{r}{R}\right)^2\left(\frac{r}{R} - 2\,\cos\phi\right)^2 + \ldots\right] \\
&\approx \frac{1}{R}\left[1 - \frac{1}{2}\left(\frac{r}{R}\right)^2 + \frac{r}{R}\cos\phi + \frac{3}{2}\left(\frac{r}{R}\right)^2\cos^2\phi\right] \\
&= \frac{1}{R}\left[1 + \frac{r}{R}\cos\phi + \frac{1}{2}\left(\frac{r}{R}\right)^2\left(3\cos^2\phi - 1\right)\right]
\end{aligned} \tag{202}$$

For the $\mathbf{r} \cdot \mathbf{R}$-term, the scalar product gives

$$\mathbf{r} \cdot \mathbf{R} = r\,R\,\cos\phi$$

so

$$\frac{\mathbf{r} \cdot \mathbf{R}}{R^3} = \frac{r}{R^2}\cos\phi \tag{203}$$

By combining (199), (202) and (203), one obtains the tidal potential expressed in terms of the zenith angle ϕ, **/3.8/**

$$\boxed{\Omega = -\frac{3}{2}Gm_l\frac{r^2}{R^3}\left(\cos^2\phi - \frac{1}{3}\right)} \tag{204}$$

13.3.1 Radial and tangential tidal forces per unit mass

Once the tidal potential Ω is determined, the radial and tangential tidal forces per unit mass can be determined from (185). For the gradient operator, we use the spherical (zenith) coordinate system shown in the right panel of Fig. 34. In this system, the gradient operator is given by (503).

The radial component $F_{t,r}$ of the tidal force per unit mass is then

$$F_{t,r} = -\frac{\partial \Omega}{\partial r} = G m_l \frac{r}{R^3}\left(3\cos^2\phi - 1\right) \tag{205}$$

or, by introducing the gravitational acceleration g from (12), **/3.9/**

$$\boxed{F_{t,r} = g\frac{m_l}{m_e}\left(\frac{r}{R}\right)^3\left(3\cos^2\phi - 1\right)} \tag{206}$$

$F_{t,r}$ is directed outward, i.e., in the $\mathbf{e}_r$-direction.

Similarly, the tangential (horizontal) component $F_{t,h}$ of the tidal force per unit mass is

$$F_{t,h} = -\frac{1}{r}\frac{\partial \Omega}{\partial r} = 3G m_l \frac{r}{R^3}\cos\phi\sin\phi \tag{207}$$

or, with $\sin 2\phi = 2\sin\phi\cos\phi$ and by introducing the gravitational acceleration g from (12), **Below /3.9/**

$$\boxed{F_{t,h} = \frac{3}{2}g\frac{m_l}{m_e}\left(\frac{r}{R}\right)^3\sin 2\phi} \tag{208}$$

$F_{t,h}$ is oriented in the $\mathbf{e}_\phi$ direction, i.e., towards the centre line connecting the centres of the Earth and the Moon (see Fig. 4).

$F_{t,r}$ and $F_{t,h}$ are identical to the tidal acceleration in (31) and (29), demonstrating the consistency of the direct method presented in Sec. 5 and the method of tidal potential of this section.

14 Introducing the tidal potential in the primitive momentum equations

The tidal forcing of the ocean can be introduced by adding the tidal forcing per unit mass (184) to the 3-dimensional momentum equation (175).

The tidal forcing on the Earth's surface Ω_t caused by the Moon (Ω_l) and the Sun (Ω_s) is

$$\Omega_t = \Omega_l + \Omega_s \tag{209}$$

Ω_l and Ω_s are both on the from of (204), so

$$\Omega_t = -\frac{3}{2}\,g\,\frac{r^4}{m_e}\left[\frac{m_l}{R_l^3}\left(\cos^2\phi_l - \frac{1}{3}\right) + \frac{m_s}{R_s^3}\left(\cos^2\phi_s - \frac{1}{3}\right)\right] \tag{210}$$

where $G = g\,r^2/m_e$ from (12) has been used.

The resulting momentum equation becomes

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + f\hat{\mathbf{z}} \times \mathbf{u} = -g\,\hat{\mathbf{z}} - \nabla(\Omega_t) + \mathcal{F} \tag{211}$$

Alternatively, the gravity term can be expressed as a gradient of a potential

$$-g\,\hat{\mathbf{z}} = -\nabla\Omega_g \tag{212}$$

where $\Omega_g = g\,z$. It is therefore common to put (211) in the form

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + f\hat{\mathbf{z}} \times \mathbf{u} = -\nabla(\Omega_g + \Omega_t) + \mathcal{F} \tag{213}$$

Note that Ω_t is a function of position and time whereas Ω_g can be considered time-independent.

Equation (211) or (213), together with the continuity equation, is used to simulate the full 3-dimensional flow of the ocean tide. The linearised shallow water equivalent of the above equations is given by (179) and (180).

15 Including the effect of elliptic orbits

For the theory presented, the distances between the Sun, the Earth and the Moon have been kept constant, implying circular orbits. This assumption is correct to lowest order, but smaller — but still non-negligible — contributions to the tide-generating forces emerge as a result of the elliptic orbit of the Earth around the Sun, and the elliptic orbit of the Moon around the Earth. Kepler's laws are key to both understand and to quantify these contributions. We first consider the Sun-Earth system.

A side note by J. Kepler, in reference to his own work[15]: *If anyone thinks that the obscurity of this presentation arises from the perplexity of my mind, ... I urge any such person to read the Conics of Apollonius. He will see that there are some matters which no mind, however gifted, can present in such a way as to be understood in a cursory reading. There is need of meditation, and a close thinking through of what is said.*

15.1 The Sun-Earth system[16]

The orbit of the Earth about the Sun is an ellipse according to Kepler's first law. The configuration is shown in Fig. 19.

Newton's law of universal gravitation states that there is a gravitational force $\mathbf{F}_e$ on the Earth from the Sun given by (see Section 5.4)

$$\mathbf{F}_e = -\frac{GM_s m_e}{R_s^2}\frac{\mathbf{R}_s}{R_s} \tag{214}$$

Newton's second law relates $\mathbf{F}_e$ (acting on the Earth with mass m_e) and acceleration

$$\mathbf{F}_e = m_e\,\frac{d^2\mathbf{R}_s}{dt^2} \tag{215}$$

Combined, expressions (214) and (215) give

$$\frac{d^2\mathbf{R}_s}{dt^2} = -\frac{GM_s}{R_s^3}\,\mathbf{R}_s \tag{216}$$

Expressed in polar coordinates (with the position of the Sun as origin), gives the following unit vectors in the directions given by the vector $\mathbf{R}_s$ (denoted $\mathbf{e}_R$) and in the direction of the angle K (hereafter $\mathbf{e}_K$; with the property $\mathbf{e}_K \perp \mathbf{e}_s$)

$$\mathbf{e}_R = (\cos K, \sin K) \tag{217}$$

$$\mathbf{e}_K = (-\sin K, \cos K) \tag{218}$$

Consequently,

$$\mathbf{R}_s = R_s\,\mathbf{e}_R \tag{219}$$

Noting that both $\mathbf{e}_R$ and $\mathbf{e}_K$ change in time and denoting the individual time derivatives with a dot, we get

$$\dot{\mathbf{e}}_R = \dot{K}\,(-\sin K, \cos K) = \dot{K}\,\mathbf{e}_K \tag{220}$$

$$\dot{\mathbf{e}}_K = -\dot{K}\,(\cos K, \sin K) = -\dot{K}\,\mathbf{e}_R \tag{221}$$

[15] E.g. `https://larouchepub.com/eiw/public/2001/eirv28n45-20011123/eirv28n45-20011123_062-keplers_optics_passion_for_scien.pdf`

[16] Mainly based on R. Fitzpatrick (): *An Introduction to Celestial Mechanics*, M. Hendershott (): *Lecture 1: Introduction to ocean tides*, and Murray & Dermott (1999), *Solar Syatem Dynamics*, Chap. 2.

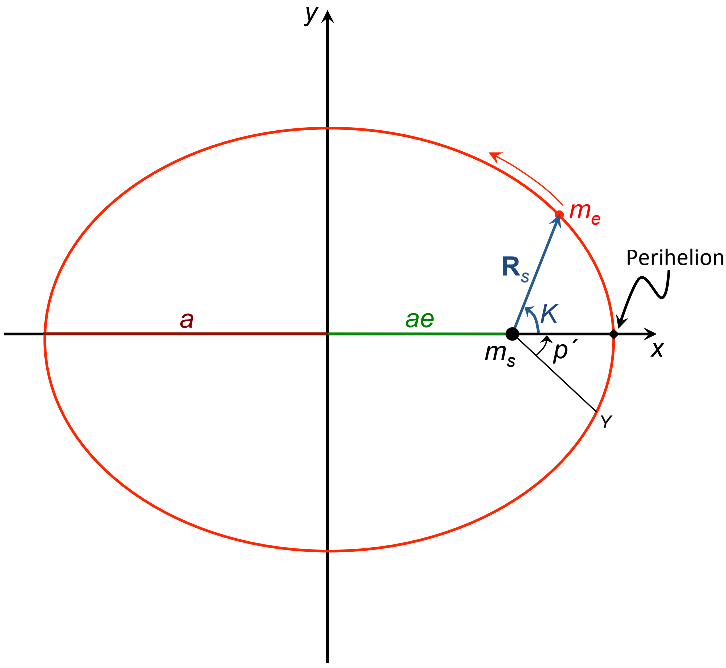

Figure 19: Illustration of the Sun-Earth system. The Earth of mass m_e orbits the Sun with mass m_s (and assumed to be fixed in space). $\mathbf{R_s}$ is the radius vector from the Sun to the Earth; K is the *true anomaly*; a is the semi-major axis; e is the eccentricity; and p' is the *longitude of the perihelion*, or the angle between Υ (the point of reference for the Earth-Sun system) and the positive x-axis. At present, $p' \approx 77°$.

Consequently,

$$\mathbf{v} = \frac{d\mathbf{R}_s}{dt} = \dot{R}_s\,\mathbf{e}_R + R_s\,\dot{\mathbf{e}}_R \tag{222}$$

$$= \dot{R}_s\,\mathbf{e}_R + R_s\dot{K}\,\mathbf{e}_K \tag{223}$$

Furthermore,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{R}_s}{dt^2} = \ddot{R}_s\,\mathbf{e}_R + \dot{R}_s\,\dot{\mathbf{e}}_R + (\dot{R}_s\dot{K} + R_s\ddot{K})\,\mathbf{e}_K + R_s\dot{K}\,\dot{\mathbf{e}}_K \tag{224}$$

or, by means of (220) and (221),

$$\mathbf{a} = (\ddot{R}_s - R_s\dot{K}^2)\,\mathbf{e}_R + (R_s\ddot{K} + 2\dot{R}_s\dot{K})\,\mathbf{e}_K \tag{225}$$

The equation of motion of the Earth, under the gravitational influence of the Sun (expression 214), is therefore given by

$$\mathbf{a} = (\ddot{R}_s - R_s\dot{K}^2)\,\mathbf{e}_R + (R_s\ddot{K} + 2\dot{R}_s\dot{K})\,\mathbf{e}_K = -\frac{GM_s}{R_s^2}\,\mathbf{e}_R \tag{226}$$

From the right-most equality, the radial component of the equation of motion becomes

$$\ddot{R}_s - R_s\dot{K}^2 = -\frac{GM_s}{R_s^2} \tag{227}$$

and the tangential component is

$$R_s\ddot{K} + 2\dot{R}_s\dot{K} = 0 \tag{228}$$

15.2 Kepler's second law

The tangential component (228) is equivalent to

$$\frac{d(R_s^2\dot{K})}{dt} = 0 \tag{229}$$

implying that the quantity

$$R_s^2\dot{K} \equiv h' \tag{230}$$

is *independent of time*.

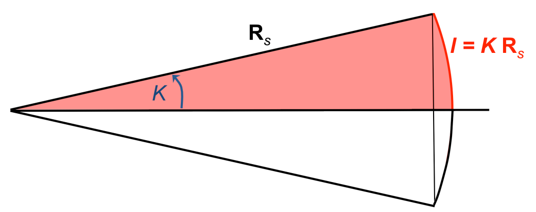

Figure 20: Illustration of the Sun-Earth system. The Earth of mass m_e orbits the Sun with mass m_s (and assumed to be fixed in space). $\mathbf{R_s}$ is the vector from the Sun to the Earth; K is the true anomaly; a is the semi-major axis; and e is the eccentricity.

The above quantity has both a geometrical and physical interpretation. The geometrical interpretation follows from Fig. 20. Per definition, the length l of the outlined (red coloured) arc is given by $l = KR_s$. For small K, or for small changes in K, the area A of the outlined sector is given by the half the base (R_s) times its height (l), i.e.,

$$A = \frac{1}{2}KR_s^2 \tag{231}$$

Temporal changes of A are thus given by

$$\frac{dA}{dt} = \frac{1}{2}\frac{d(KR_s^2)}{dt} = \frac{1}{2}\frac{dh'}{dt} = 0 \tag{232}$$

Therefore, A is constant in time. Thus, Kepler's second law) states that the radius vector from the Sun to the Earth (or to any planet) sweeps out equal areas in equal times. **/Fig. 3.9/**

15.3 Kepler's first law

The radial component (227) is non-linear in R_s, but it can be turned into a linear equation by introducing

$$u = \frac{1}{R_s} \quad \text{or} \quad R_s = \frac{1}{u} \tag{233}$$

Temporal derivatives of R_s can now be expressed in terms of $u = u(K(t))$ by repeated use of the chain rule

$$\dot{R}_s = -\frac{1}{u^2}\dot{u} = -\frac{1}{u^2}\frac{du}{dK}\frac{dK}{dt} = -h'\frac{du}{dK} \tag{234}$$

where h' is given by (230). In a similar manner,

$$\ddot{R}_s = -h'\frac{d^2u}{dK^2}\frac{dK}{dt} = -u^2(h')^2\frac{d^2u}{dK^2} \tag{235}$$

Inserting into (227) leads to

$$-(h')^2u^2\frac{d^2u}{dK^2} - (h')^2u^3 = -u^2GM_s \tag{236}$$

or

$$\frac{d^2u}{dK^2} + u = \frac{GM_s}{(h')^2} \tag{237}$$

The homogeneous solution to the above equation is proportional to $\cos K$. Including the constant right-hand side of (237) to the solution gives, with $u = 1/R_s$:

$$\frac{1}{R_s} = A'\cos K + \frac{GM_s}{(h')^2} \tag{238}$$

where A' is a constant of integration. A second integration constant is embedded in the argument of the cosine-function, so that the argument K can be adjusted by a constant phase shift according to the chosen coordinate system.

Expression (238) is, as expected, the parametric equation of an ellipse, see appendix D. By direct comparison with (541), equation (238) can be written in terms of the ellipse's semi-major axis a and the eccentricity e:

$$A' = \frac{e}{a(1-e^2)} \tag{239}$$

and

$$\frac{GM_s}{(h')^2} = \frac{1}{a(1-e^2)} \tag{240}$$

From expression (238), Kepler's first law states that planets move in ellipses with the Sun at one focus.

15.4 Kepler's third law

During one orbital period T_K, the area A swept out by $\mathbf{R}_s$ is

$$A = \pi ab \tag{241}$$

From expression (232), integrating over one period,

$$A = \frac{h\,T_K}{2} \tag{242}$$

Therefore,

$$T_K^2 = 4\pi^2\,\frac{a^2\,b^2}{(h')^2} = 4\pi^2\,\frac{a^4\,(1-e^2)}{(h')^2} \tag{243}$$

where use has been made of expression (518) in the last equality.

Furthermore, from (240),

$$(h')^2 = a(1-e^2)\,GM_s \tag{244}$$

By combining (243) and (244), we obtain) **/3.14/**

$$\boxed{T_K = 2\pi\sqrt{\frac{a^3}{GM_s}}} \tag{245}$$

The corresponding angle frequency $\omega_K = 2\pi/T_K$ becomes

$$\omega_K = \sqrt{\frac{GM_s}{a^3}} \tag{246}$$

Note that T_K, and therefore ω_K, is only dependent of the elliptic parameter a and the environmental parameters G and M_s.

Kepler's third law , from expression (245), states that the square of the orbital period of a planet is proportional to the cube of its semi-major axis.

15.5 Mean anomaly E_0

The true anomaly K is 2π-periodic, but it does not change linearly in time for $e \neq 0$. It is, however, *convenient* to consider a 2π-periodic angle that changes linearly in time. This parameter — named the *mean anomaly* and denoted E_0 — is defined by

$$E_0 = \omega_K\,t \tag{247}$$

with ω_K coming from expression (246).

From `https://en.wikipedia.org/wiki/Mean_anomaly`: *The mean anomaly does not measure an angle between physical objects. It is simply a convenient, uniform measure of how far around its orbit a body has progressed since perihelion. The mean anomaly is one of three angular parameters (known historically as 'anomalies') that define a position along an orbit, the other two being the eccentric anomaly and the true anomaly.*

15.6 Real vs mean length of the radius vector, and Kepler's equation

It is mathematically convenient to introduce the eccentric anomaly E (see Fig. 21), in stead of the true anomaly K, in expression (238). The task is therefore to relate the elliptic 'radius' R_s with that of the reference circle $\overline{R}_s$, for brevity a, with respect to E.

We start with deriving an expression relating R_s and a, followed by Kepler's equation giving an equation for E.

15.6.1 R_s vs $\overline{R}_s$

Geometric interpretation of the left panel in Fig. 21 gives

$$FA = 0A - 0F = a\cos E - ae = a(\cos E - e) \tag{248}$$

and

$$FA = R_s \cos K \tag{249}$$

The combination of the above expressions gives

$$R_s = \frac{a(\cos E - e)}{\cos K} \tag{250}$$

or, alternatively,

$$\cos K = \frac{a(\cos E - e)}{R_s} \tag{251}$$

The equation for an ellipse in polar coordinates is given by expression (540)

$$R_s = \frac{a(1-e^2)}{1+e\,\cos K} \tag{252}$$

or

$$R_s(1+e\,\cos K) = a(1-e^2) \tag{253}$$

Insertion of (251) yields

$$R_s\left(1+e\,\frac{a(\cos E - e)}{R_s}\right) = a(1-e^2) \tag{254}$$

or, after solving for R_s,

$$R_s = a(1-e\cos E) \tag{255}$$

In stead of the circle radius a, we may use $\overline{R}_s$ — the mean and constant radius of the principal circle — leading to

$$R_s = \overline{R}_s(1-e\cos E) \tag{256}$$

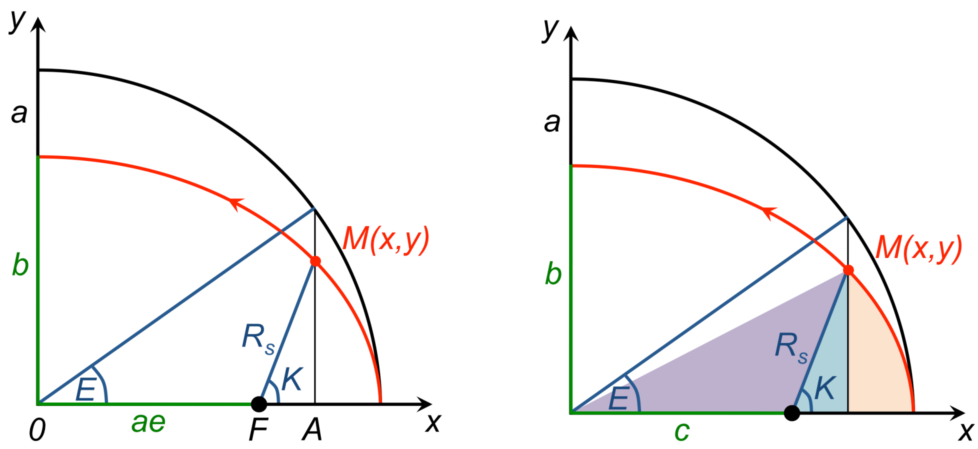

Figure 21: Illustration of the Sun-Earth system. The Earth of mass m_e orbits the Sun with mass m_s (and assumed to be fixed in space). $\mathbf{R_s}$ is the vector from the Sun to the Earth; K is the true anomaly; a is the semi-major axis; and e is the eccentricity. The colours in the right panel are used in th ederivation of Kepler's equation in Sec. 15.6.2.

The non-circularity of the elliptic orbit is thus given by

$$\frac{\overline{R}_s}{R_s} = \frac{1}{1 - e\cos E} \tag{257}$$

For small values of the eccentricity e, the right-hand side of (257) can be series expanded by means of the binomial theorem, see appendix H.1. We obtain, to lowest order

$$\frac{\overline{R}_s}{R_s} = 1 + e\cos E \tag{258}$$

Next, we need to obtain an equation for E following the orbiting motion of the body M in Fig. 21.

15.6.2 Kepler's equation

The area swept out by the orbiting body M can be interpreted from the right panel of Fig. 21.

Firstly, the area A_E defined by the eccentric anomaly and the reference circle is given by

$$A_E = \pi\, a^2\, \frac{E}{2\pi} = E\frac{a^2}{2} \tag{259}$$

where E is in radians. Any y-value on an ellipse equals the constant factor

$$b/a = \sqrt{1-e^2} \tag{260}$$

of the corresponding y-value on the reference circle (see appendix D.3). The area of the tri-coloured region A_{col} in Fig. 21 therefore equals

$$A_{\text{col}} = A_E\,\sqrt{1-e^2} = E\frac{a^2\sqrt{1-e^2}}{2} \tag{261}$$

The sector area swept out by K is given by area $A_{\rm col}$ minus the pink triangle in Fig. 21, $A_{\rm pink}$. The latter equals half of the baseline $c = ae$ times the height $a\sin E\sqrt{1-e^2}$:

$$A_{\rm pink} = a^2 e \sin E \frac{\sqrt{1-e^2}}{2} \tag{262}$$

The sought-after sector area A_K swept out by K is therefore given by

$$A_K = A_{\rm col} - A_{\rm pink} = a^2 \frac{\sqrt{1-e^2}}{2}(E - e\sin E) \tag{263}$$

The area A_K is proportional to time t, the latter set to zero at *perihelion*[17], i.e., the point of least distance between the Sun and the Earth or, which is equivalent, when M crosses the positive x-axis in Fig. 21. Therefore,

$$A_K = C\,t \tag{264}$$

where C is a constant to be determined.

Let T_K and $\omega_{\rm K}$ denote the period and frequency of the elliptic motion, respectively. When $E = 2\pi$, A_K equals the area of the ellipse,

$$A_K = \pi ab = \pi a^2\sqrt{1-e^2} \tag{265}$$

where the relationship (519) has been used in the last equality. Consequently,

$$C\,T_K = \pi a^2\sqrt{1-e^2} \tag{266}$$

so

$$C = \pi a^2 \frac{\sqrt{1-e^2}}{T_K} \tag{267}$$

By combining expressions (263), (264) and (267), we get Kepler's equation

$$E - e\sin E = \omega_{\rm K}\,t \tag{268}$$

yielding a relationship between the eccentric anomaly E and the time t since perihelion.

Introducing the mean anomaly from expression (247) gives Kepler's equation

$$E_0 = E - e\sin E \tag{269}$$

15.6.3 Solving Kepler's equation

Kepler's equation (269) is transcendental and cannot be solved directly. But the equation can be solved by iteration. As an example, equation (269) can be put in the form

$$E_{i+1} = E_0 + e\sin E_i\,, \qquad i = 0, 1, ... \tag{270}$$

Since the magnitude of the E_i derivative of the right-hand side is less than unity, $\max|e\cos E_i| = e < 1$, the given iterative scheme converge (the Fixed Point Theorem).

[17] For the Earth-Moon system, the corresponding point is called *perigee*.

For the first approximations, this give

$$E_1 = E_0 + e\sin E_0 \quad (271)$$
$$E_2 = E_0 + e\sin E_1 = E_0 + e\sin(E_0 + e\sin E_0) \quad (272)$$

The last term on the right-hand side of (272) can be expanded by means of the sum and difference identity (619), leading to

$$E_2 = E_0 + e\left[\sin E_0 \cos(e\sin E_0) + \cos E_0 \sin(e\sin E_0)\right] \quad (273)$$

For small arguments x, the following series expansions are valid

$$\sin x = x + \mathcal{O}(x^3) \quad (274)$$
$$\cos x = 1 - \frac{x^2}{2} + \mathcal{O}(x^4) \quad (275)$$

Thus,

$$E_2 = E_0 + e\left[\sin E_0 + e\sin E_0 \cos E_0\right] + \mathcal{O}(e^3) = E_0 + e\,\sin E_0 + \mathcal{O}(e^2) \quad (276)$$

For an example of how to numerically determine E, see e.g.
`https://www.csun.edu/~hcmth017/master/node16.html`.

15.6.4 $\overline{R}_s/R_s$ expressed by means of E_0

Similar to the procedure outlined in Section 15.6.3, the ratio $\overline{R}_s/R_s$, from (258), can be expanded as follows:

$$\begin{aligned}\frac{\overline{R}_s}{R_s} &= 1 + e\sin E \\ &\approx 1 + e[\cos(E_0 + e\sin E_0)] \\ &= 1 + e[\cos E_0 \cos(e\sin E_0) - \sin E_0 \sin(e\sin E_0)] \\ &= 1 + e\cos E_0 + \mathcal{O}(e^2) \quad (277)\end{aligned}$$

Thus, to first order in e,

$$\frac{\overline{R}_s}{R_s} = 1 + e\cos E_0 \quad (278)$$

15.6.5 Introducing the ecliptic longitude and the longitude of perihelion

The ratio between the length of the actual and mean radius vector in equation (258) can be expressed in terms of the Sun's *geocentric mean ecliptic longitude* h, increasing by $0.0411°$ per mean solar hour, and the *longitude of the solar perigee* (called *perihelion*) p'. From Fig. 22, the *celestial longitude* λ_s, is

$$\lambda_s = p' + K \quad (279)$$

By defining the *mean ecliptic longitude* relative to Υ according to

$$h = p' + E_0 \quad (280)$$

it follows that

$$E_0 = h - p' \quad (281)$$

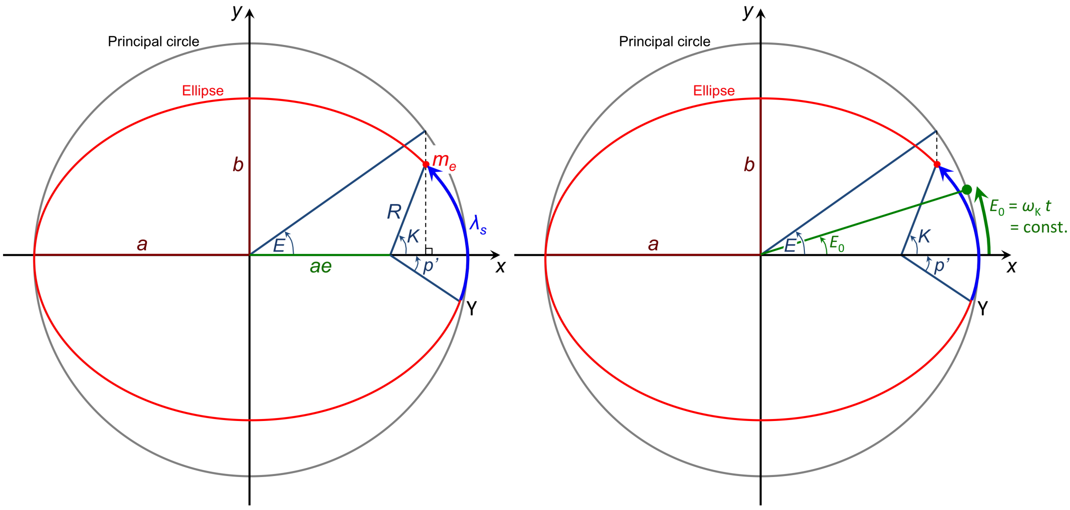

Figure 22: Left panel is as Fig. 19, but including the principal circle; the *eccentric anomaly* E (the projection of the Earth, m_e, onto the principal circle in the y-direction); and the *celestial longitude* (or *true longitude*) λ_s, running from ♈ to the actual position of the Earth. In the right panel, an imaginary body (green dot) describes a circular orbit with constant rotation rate ω_{K} as defined on page 65.

Thus, from (278), **/3.15/**

$$\boxed{\frac{\overline{R}_s}{R_s} = 1 + e\cos(h - p')} \tag{282}$$

Furthermore, by combining (290) and (280), we have the relationship

$$\lambda_s = h + K - E_0 \tag{283}$$

In the above expression and elsewhere, the true anomaly K can be written in terms of E, and subsequently E_0, by combining the two equivalent expressions (252) and (257):

$$\frac{1 + e\cos K}{1 - e^2} = \frac{1}{1 - e\cos E} \tag{284}$$

The smallness of e leads to, by means of the binomial theorem (614) applied to (284), a series expansion in e:

$$\begin{aligned} 1 + e\cos K = \frac{1 - e^2}{1 - e\cos E} &= (1 - e^2)(1 + e\cos E + e^2\cos^2 E + \mathcal{O}(e^3)) \\ &= 1 + e\cos E + e^2\cos^2 E - e^2 + \mathcal{O}(e^3) \end{aligned} \tag{285}$$

Consequently, to first order in e,

$$\cos K = \cos E + e\,(\cos^2 E - 1) = \cos E + e\,\sin^2 E \tag{286}$$

Upon substituting expression (271) for E and following the procedure outlined in Section 15.6.3, we get

$$\begin{aligned}\cos K &= \cos(E_0 + e\sin E_0) - e\sin^2(E_0 + e\sin E_0) \\ &= \cos E_0 \cos(e\sin E_0) - \sin E_0 \sin(e\sin E_0) \\ &\qquad - e\,[\sin E_0 \cos(e\sin E_0) + \cos E_0 \sin(e\sin E_0)]^2 \\ &= \cos E_0 - e\sin^2 E_0 - e\,[\sin E_0 + e\sin E_0 \cos E_0]^2 + \mathcal{O}(e^2) \\ &= \cos E_0 - 2e\,\sin^2 E_0 + \mathcal{O}(e^2) \qquad (287)\end{aligned}$$

Similarly, by applying the procedure outlined in Section 15.6.3,

$$\begin{aligned}\cos(E_0 + 2e\,\sin E_0) &= \cos E_0 \cos(2e\sin E_0) - \sin E_0 \sin(2e\sin E_0) \\ &= \cos E_0 - \sin E_0\, 2e\sin E_0 + \mathcal{O}(e^2) \\ &= \cos E_0 - 2e\,\sin^2 E_0 + \mathcal{O}(e^2) \qquad (288)\end{aligned}$$

The right-hand side of the two expressions (287) and (288) are identical to the first order in e, implying that (to the first order in e)

$$K = E_0 + 2e\,\sin E_0 \qquad (289)$$

Thus, fom expression (283),

$$\lambda_s = h + 2e\,\sin E_0 \qquad (290)$$

or, by using (281), **/3.16/**

$$\boxed{\lambda_s = h + 2e\,\sin(h - p')} \qquad (291)$$

15.7 Right Ascension

The cotangent rule (642) can be applied to the spherical triangle $\Upsilon SS'$ in Fig. 23. This gives

$$\cos A_s \cos\epsilon_s = \sin A_s \cot\lambda_s \qquad (292)$$

where we have used that the angle at S' is a right-angle. The above equation is equivalent to

$$\sin A_s \cos\lambda_s = \sin\lambda_s \cos A_s \cos\epsilon_s \qquad (293)$$

For $\cos\epsilon_s$, the half-angle identity (633) can be used:

$$\cos\epsilon_s = \frac{1 - \tan^2(\epsilon_s/2)}{1 + \tan^2(\epsilon_s/2)} \qquad (294)$$

Thus,

$$\sin A_s \cos\lambda_s \left(1 + \tan^2\frac{\epsilon_s}{2}\right) = \sin\lambda_s \cos A_s \left(1 - \tan^2\frac{\epsilon_s}{2}\right) \qquad (295)$$

Rearranging gives

$$\sin A_s \cos\lambda_s - \cos A_s \sin\lambda_s = -(\sin A_s \cos\lambda_s + \sin\lambda_s \cos A_s)\tan^2\frac{\epsilon_s}{2} \qquad (296)$$

or, by means of the sum and difference identity (619),

$$\sin(A_s - \lambda_s) = -\tan^2\left(\frac{\epsilon_s}{2}\right)\sin(A_s + \lambda_s) \qquad (297)$$

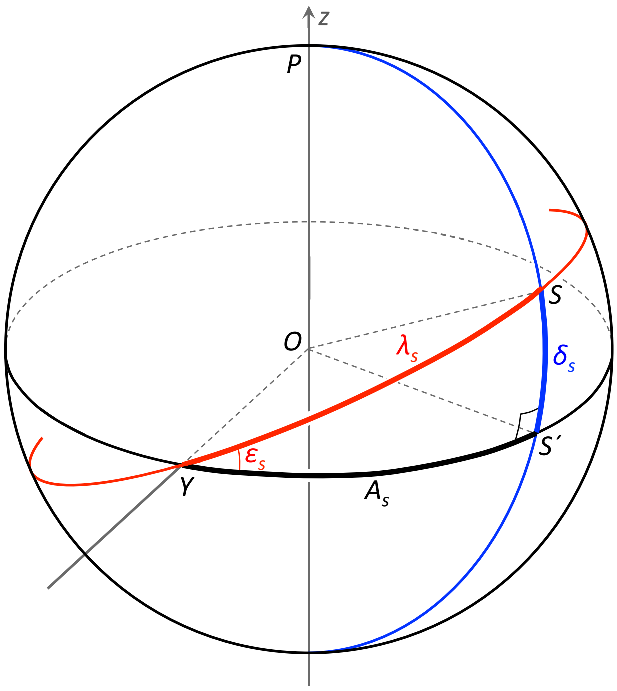

Figure 23: The celestial sphere with the Vernal Equinox (Υ), the Sun (S), the equatorial plane $O\Upsilon S'$, the ecliptic $O\Upsilon S$, the celestial north pole P , and the meridian PSS' . In the spherical triangle $\Upsilon SS'$, the arc ΥS is the celestial longitude λ_s , the arc $\Upsilon S'$ is the Right Ascension A_s , and the arc SS' is the declination δ_s . The angle at Υ is the obliquity ϵ_s and the angle at S' is a right-angle. Caption and drawing is based on Fig. 7.7 in Capderou (2004).

Since $\epsilon_s = 0.41$ is relatively small, $A_s \approx \lambda_s$. Series expansion to lowest order gives, from expression (639): **/3.17/**

$$\boxed{A_s = \lambda_s - \tan^2\left(\frac{\epsilon_s}{2}\right)\sin(2\lambda_s)} \tag{298}$$

15.8 Equation of time

The time given on a sundial does approximately, but not exactly, match the time on the clock. The two time frames would be identical if Earth's orbit was circular and if Earth's equatorial plane was identical to the plane swept out by Earth's orbit. In this case Earth's motion could be described with a constant, circular rotation rate. Actually, the clock time is defined in this way: That any given time interval represents the same (circular, ecliptic) motion of the Earth around the Sun. From a geocentric perspective, the latter represents the motion of the *mean Sun*, where *mean* represents averaging over a year, i.e., averaging over one full orbit.

The motion of the *true Sun* takes into account the elliptic path (Fig. 19), leading to alternating phases of acceleration and deceleration during the orbit (Section 15.2). Furthermore, the tilt between the celestial longitude λ_s and the celestial equator A_s (Fig. 23), implies that different distances are travelled during the same time interval. The difference between the Right Ascension of the mean Sun (clock time) and the Right Ascension of the true Sun (sundial time), at any time t, is known as the *equation of time*.

15.8.1 Equation of center

The effect of the Earth's acceleration/deceleration along it's elliptical orbit relative to a circular orbit with constant rotation rate is called the *equation of center*, E_C. Based on this, E_C is the difference between the true anomaly K and the mean anomaly E_0, see Fig. 22:

$$E_{\mathrm{C}} = K - E_0 \tag{299}$$

Thus, by means of (289) and to the first order in e,

$$E_{\mathrm{C}} = 2e\,\sin E_0 \tag{300}$$

$E_{\mathrm{C}} = \omega_{\mathrm{K}}\,t$ has, per construction, a period of one year, with $t = 0$ at perihelion. E_{C} is positive for the first half of the year, implying that the actual position of Earth is ahead of an imaginary body moving at constant speed along the reference circle. E_{C} has an amplitude of

$$E_{\mathrm{C,amp}} = 2e = 0.0334\ \mathrm{rad} \tag{301}$$

15.8.2 Longitudinal differences

The difference in length E_R between the Right Ascension A_s and the celestial longitude λ_s, see Fig. 23, is

$$E_{\mathrm{R}} = A_s - \lambda_s = -\tan^2\frac{\epsilon_s}{2}\sin(2\lambda_s) \tag{302}$$

where use has been made of expression (298) in the last equality.

Realtive to perihelion, the argument

For small eccentricity, from (290),

$$\lambda_s \approx h = E_0 + p' \tag{303}$$

where (280) has been used in the last equality. Thus,

$$E_{\rm R} = -\tan^2\frac{\epsilon_s}{2}\sin 2(E_0 + p') \tag{304}$$

15.8.3 Equation of time, final expression

Based on the above, the expression for the equation of time $E_{\rm T}$, can be put in the form

$$E_{\rm T} = E_{\rm C} + E_{\rm R} = 2e\,\sin E_0 - \tan^2\frac{\epsilon_s}{2}\sin 2(E_0 + p') \tag{305}$$

with $E_0 = \omega_{\rm K}\,t$ from (247) and $p' = 77^\circ = 1.3439$ rad.

15.8.4 Equation of time, graphical representation

If we let t denote the day D of the year and let $t = 0$ correspond to January 1, the passage at perigee on January 3 occurs at $D = 3$. With these choices, and using that a full year has 365 days (for simplicity), we get

$$E_{\rm C} = 2e\,\sin\left[\frac{2\pi}{365}(D-3)\right] \tag{306}$$

$E_{\rm C}$ is in radians, but we want to express it in suitable time units. This can be done by transferring the amplitude $2\,e$ from radians to days, and thereafter to minutes, by using the relationship

$$2\,\pi \quad \text{corresponds to} \quad 1\ D = 1440\ \text{min} \tag{307}$$

With $e = 0.0167$, we get

$$E_{\rm C}[\text{min}] = 0.0334\frac{1440}{2\,\pi}\,\sin\left[\frac{2\pi}{365}(D-3)\right] = 7.44\,\sin\left[\frac{2\pi}{365}(D-3)\right] \tag{308}$$

where [min] denotes that $E_{\rm C}$ is given in minutes.

Similarly, the argument $2\,E_0$ in $E_{\rm R}$ becomes

$$2\,E_0 = 2\,\frac{2\pi}{365}(D-3) \tag{309}$$

The second argument in $E_{\rm R}$, p', is a constant phase relative to perigee, see Fig. 22. $p' = -77^\circ$ or, in radians,

$$p' = -77\,\frac{2\,\pi}{360} \tag{310}$$

Thus,

$$E_{\rm R} = -\tan^2\frac{\epsilon_s}{2}\,\sin\left[4\,\pi\left(\frac{D-3}{365} - \frac{77}{360}\right)\right] \tag{311}$$

With $\epsilon_s = 23^\circ\ 27' = 23.45^\circ$, we obtain

$$E_{\rm R}[\text{min}] = -9.87\,\sin\left[4\,\pi\left(\frac{D-3}{365} - \frac{77}{360}\right)\right] = -9.87\,\sin\left[\frac{4\,\pi}{365}(D-81.07)\right] \tag{312}$$

In summary,

$$E_{\rm T}[\text{min}] = 7.44\,\sin\left[\frac{2\pi}{365}(D-3)\right] - 9.87\,\sin\left[\frac{4\,\pi}{365}(D-81.07)\right] \tag{313}$$

with $E_{\rm C}$ in minutes and D in days with $D = 1, 2, ..., 365$, with $D = 1$ for January 1. The resulting variation throughout the year is shown in Fig. 24. **/Fig. 3.10/**

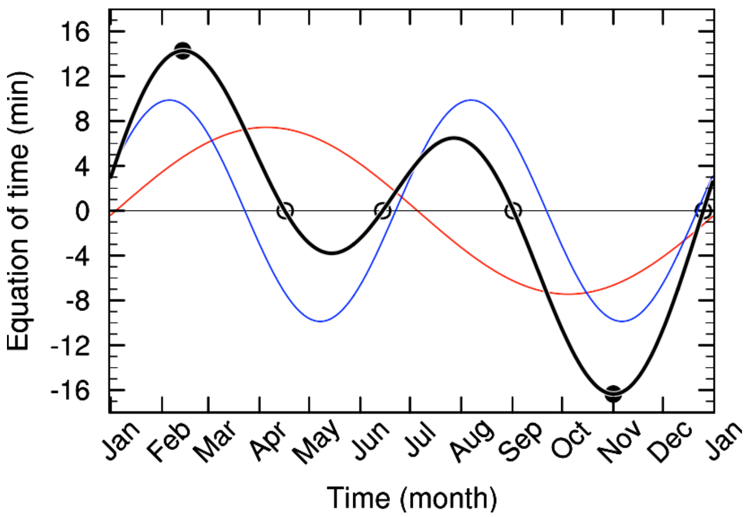

Figure 24: The equation of time E_{T} (black line), with it's two contributions E_{C} (red) and E_{R} (blue). Positive values means that the time according to a sundial is ahead of the mean time (i.e., ahead of the clock time). Filled circles show the extrema of E_{T}, with maximum value at 14 Feb with $E_{\mathrm{T}} = +14.27$ min and minimum value at 1 Nov with $E_{\mathrm{T}} = -16.35$ min. Open circles show the four zero-crossings at 16 Apr, 14 Jun, 1 Sep, and 25 Dec. The stated dates are approximate; they may vary by one to two days depending on the actual year. On longer time scales, e and ϵ_s vary slowly, implying changes in E_{T}.

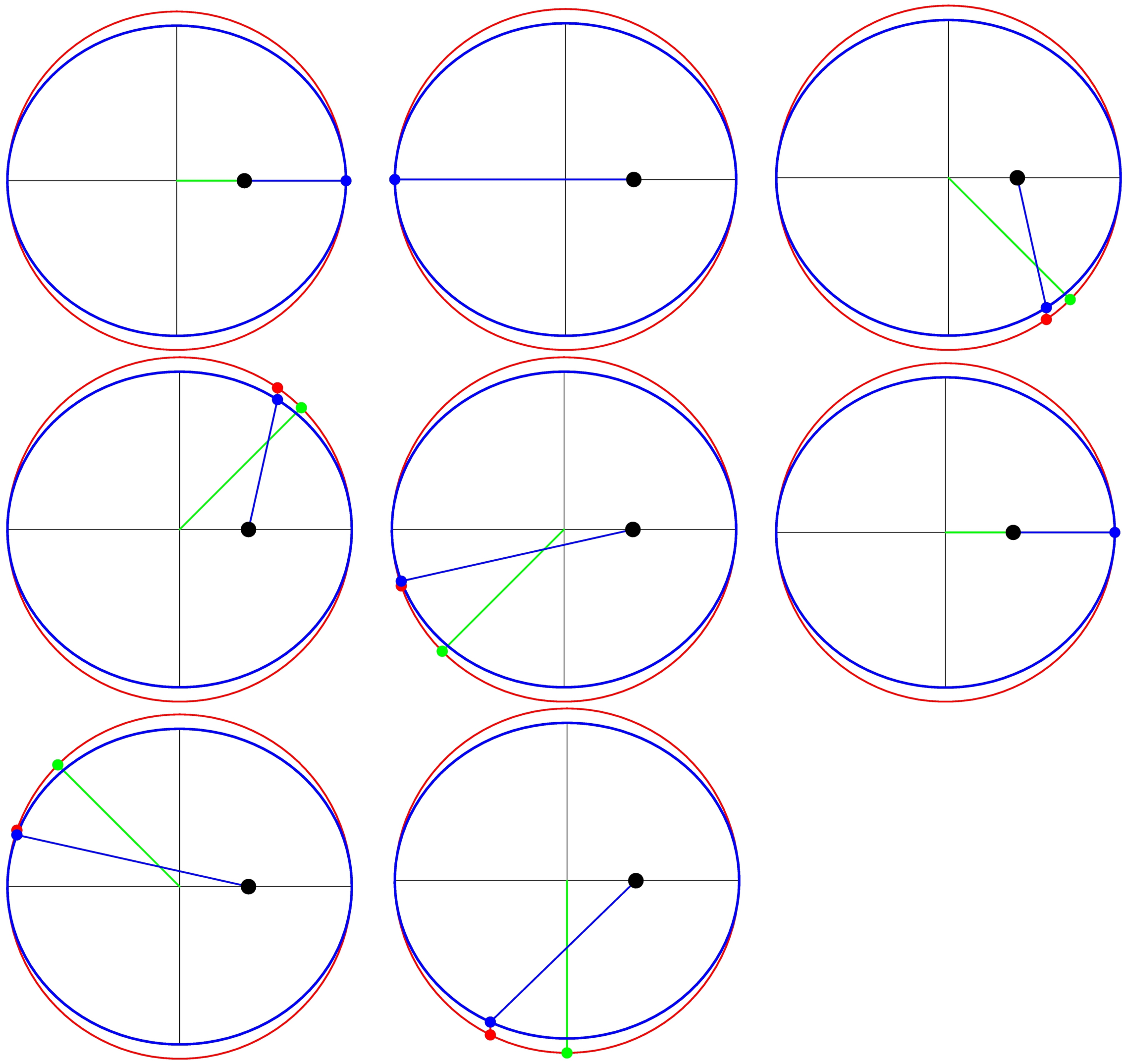

Figure 25: A sequence showing the true anomaly (blue line/dot), the true anomaly vertically projected onto the reference circle (red dot and red circle, respectively), and the mean anomaly (green line/dot) on the reference circle, during a full period T. The panels run from the upper, left panel and down, column-by-column, with increments of $T/7$. Thus, the upper, left panel is at time $t = 0$, the middle, left panel is at $t = T/7$, etc. The eccentricity is set to 0.4, so the mismatch between the true and mean anomaly is *greatly* exaggerated compared to the Earth-Sun system. An animation of the above sequence is available from `http://folk.uib.no/ngfhd/Teaching/Div/anim.gif`.

Part III

Tidal harmonic analysis

16 Background

The idea behind tidal harmonic analysis is to represent tidal generated sea surface height variations or tidal currents by means of a sum of simple trigonometric functions, the latter with frequencies given by the equilibrium theory.

Let the surface elevation ζ_m denote an arbitrary tidal constituent m /p. 12/

$$\zeta_m(t) = H_m \cos(\omega_m t - g_m) \tag{314}$$

with amplitude H_m, frequency ω_m and phase g_m. From the tidal equilibrium analysis, ω_m is known, whereas H_m and g_m are to be determined.

By means of identity (620), expression (314) can be put in the form

$$\boxed{\zeta_m(t) = A_m \cos\omega_m t + B_m \sin\omega_m t} \tag{315}$$

Here

$$A_m = H_m \cos g_m\,, \qquad \text{and} \qquad B_m = H_m \sin g_m \tag{316}$$

The sum of the square and the ratio of the the above relationships give

$$H_m = \sqrt{A_m^2 + B_m^2}\,, \qquad \text{and} \qquad g_m = \arctan\frac{B_m}{A_m} \tag{317}$$

Consequently, once A_m and B_m are known, H_m and g_m, and consequently $\zeta_m(t)$ from expression (314), are also known.

The observed (real) surface elevation $X(t)$ can always be written in terms of the mean sea level Z_0 (only slowly changing in time, for instance because of global warming, but treated as constant here), tidal variations $T(t)$, and a residual $R(t)$:

$$X(t) = Z_0 + T(t) + R(t) \tag{318}$$

By construction, $T(t)$ describes the variations in $X(t)$ better and better as more tidal constituents are included.

Let us first assume that the residual vanishes, i.e., that $R(t) = 0$. The coefficients A_m and B_m can then be determined by searching for the minimum difference between the observed sea surface anomalies, $X'(t) = X(t) - Z_0$, and the tidal contribution to the sea surface elevation, $T(t)$.

For a single tidal constituent m, the minimisation can be expressed as searching the minimum value of the non-negative quantity

$$(X'(t) - \zeta_m)^2 \tag{319}$$

For a given time interval, say from $t = 0$ to $t = t_k$, this is equivalent of minimising the integral

$$I = \int_{t=0}^{t_k} (X'(t) - \zeta_m)^2\, dt \tag{320}$$

Since $X'(t)$ is known from the observed sea level time series, the integral I is a function of A_m and B_m only (remember that ω_m is known from the equilibrium theory). We therefore search for those A_m and B_m that minimise I (see footnote[18]):

$$\frac{\partial I}{\partial A_m} = 0\,, \quad \text{and} \quad \frac{\partial I}{\partial B_m} = 0 \tag{321}$$

Upon differentiation,

$$\frac{\partial I}{\partial A_m} = -2\int_{t=0}^{t_k} (X'(t) - \zeta_m)\, \cos\omega_m t\, dt = 0 \tag{322}$$

and

$$\frac{\partial I}{\partial B_m} = -2\int_{t=0}^{t_k} (X'(t) - \zeta_m)\, \sin\omega_m t\, dt = 0 \tag{323}$$

Insertion of ζ_m from (315) gives, respectively,

$$\int_{t=0}^{t_k} X'(t)\, \cos\omega_m t\, dt - A_m \int_{t=0}^{t_k} \cos^2\omega_m t\, dt - B_m \int_{t=0}^{t_k} \sin\omega_m t\, \cos\omega_m t\, dt = 0 \tag{324}$$

and

$$\int_{t=0}^{t_k} X'(t)\, \sin\omega_m t\, dt - A_m \int_{t=0}^{t_k} \sin\omega_m t\, \cos\omega_m t\, dt - B_m \int_{t=0}^{t_k} \sin^2\omega_m t\, dt = 0 \tag{325}$$

Simplification

For simplicity, we assume that t_k equals multiple periods T_m of the tidal constituent m. In this special case, using identity (638),

$$\int_{t=0}^{t_k = nT_m} \sin\omega_m t\, \cos\omega_m t\, dt = 0 \tag{326}$$

where n is an integer.

Furthermore, from identities (636) and (637), it follows that

$$\int_{t=0}^{t_k} \cos^2\omega_m t\, dt = \int_{t=0}^{t_k} \sin^2\omega_m t\, dt = \frac{t_k}{2} \tag{327}$$

Consequently, from expressions (325) and (324),

$$A_m = \frac{2}{t_k} \int_{t=0}^{t_k = n\,T_m} X'(t)\, \cos\omega_m t\, dt \tag{328}$$

and

$$B_m = \frac{2}{t_k} \int_{t=0}^{t_k = n\,T_m} X'(t)\, \sin\omega_m t\, dt \tag{329}$$

Note that the right hand side of the above integrals are known; $X'(t)$ is observed and ω_m is known from the equilibrium theory. A_m and B_m, and consequently ζ_m from (315), can thus be determined.

[18]See Section 6 in the note by Bjørn Gjevik, available at `http://folk.uio.no/bjorng/tides_unis.pdf`.

17 Implementation, sea surface height variations

Any integral can be transferred to sums, and the sums can then readily be evaluated by numerical methods. We therefore transfer the integrals (328) and (329) to sums.

Note: For simplicity, we assume that the time series to be analysed is continuous, i.e., without interruptions or erroneous values. In case of a non-continuous time series, gaps need to be filled by means of neighbouring and/or proxy observations, theoretical considerations or *a priori* assumptions.

The transformation can, as an example, be done with the following choices/substitutions:

Length of timestep dt:

$$dt = \Delta t \tag{330}$$

In the following we use $\Delta t = 1$ hr, implying hourly readings of the sea surface elevation. Other time steps can, of course, be used.

Number of observations $K+1$:
We assume that the sea surface height anomaly $X'(t)$ consists of $K+1$ discrete (hourly) readings, so the continuous time series $X'(t)$ goes over to the discrete series

$$X'_k\,, \qquad k = 0, 1, ..., K \tag{331}$$

Timing t of the $K+1$ observations:
Based on the above, the discrete readings take place at

$$t = 0, \Delta t, ..., K\,\Delta t \tag{332}$$

The total duration t_k of the time series:

$$t_k = K\,\Delta t\,, \qquad \text{so} \qquad \frac{1}{t_k} = \frac{1}{K\,\Delta t} \tag{333}$$

The frequency ω_m and period T_m of the tidal constituent m:

$$\omega_m = \frac{2\pi}{T_m} \tag{334}$$

Based on the above, the integrals (328) and (329) can be transferred to discrete form:

$$A_m = \frac{2}{K} \sum_{k=0}^{K} X'_k \cos\left[\frac{2\pi}{T_m}\,k\,\Delta t\right] \tag{335}$$

$$B_m = \frac{2}{K} \sum_{k=0}^{K} X'_k \sin\left[\frac{2\pi}{T_m}\,k\,\Delta t\right] \tag{336}$$

In the above expressions, $1/\Delta t$ from (333) is cancelled by $dt = \Delta t$ in the integrals (328) and (329).

Once A_m and B_m are determined from the above formulas, the amplitude H_m and the phase g_m can be determined from the two expressions in (317). This, in turn, determines the tidal time series from expression (315) that approximates the observed time series $X'(t)$.

Note that the procedure is only valid when the length of the analysed portion of the sea level time series is a multiple M of the tidal period in consideration, or for $t_k = M\,T_m$, where M is a positive integer.

For accurate extraction of a given tidal constituent from the observed time series, it is important to sample over many tidal periods, so $M \gg 1$. As a rule of thumb, we should sample over a time interval that includes, at least, one spring period (14.7667 days). The reason for this is that a shorter time interval will miss the prominent interaction between the $\mathbf{M_2}$ and $\mathbf{S_2}$ tides. As a rule of thumb – and for analysis or modelling of any time series – improved accuracy is obtained by sampling at least 3–5 times longer than the period of main variation. Thus, taking into account that the semi-diurnal tidal constituents have a period of around 12 hr, it is advisable that M is larger than 100 (since $M = 30$ corresponds to 14 days for the semi-diurnal tides).

17.1 Extraction of the $\mathrm{M_2}$ tide

For $\mathbf{M_2}$, the above constraints are met, as an example, with the following choices: Firstly, from the tidal equilibrium analysis, we have that

$$T_{\mathrm{M_2}} = 12.42 \text{ hr} \tag{337}$$

With hourly sea level data, $\Delta t = 1$ hr. Since 50 times the $\mathbf{M_2}$ period is an integer, we can chose $M = 50$ (i.e., 50 times the $\mathbf{M_2}$ period), since this leads to an integer number of time steps:

$$k = 50 \times 12.42 = 621 \tag{338}$$

Thus, 621 hourly time steps (or an integer multiple thereof, see Note 2) can be used to extract the $\mathbf{M_2}$ constituent from the observed time series.

17.2 Note 1

An integer number of time steps are required for carrying out the above given sums. Therefore, as an example, $M = 30$ times the $\mathbf{M_2}$ period would not work properly since $30 \times 12.42 = 372.6$ time steps.

For the $\mathbf{M_2}$ tide, M is therefore chosen to ensure that the product $M\,T_{\mathrm{M_2}}$ is an integer. Strictly speaking, the above holds for $T_{\mathrm{M_2}}$ given with a two-digit precision. Higher precisions can, of course, be treated similarly.

17.3 Note 2

Based on the total length of the observed time series $X'(t)$, we may chose any multiple M of T_m. In general, the accuracy of our model increases with increasing M. For a time series $X'(t)$ covering a little more than 100 days, we may chose $M = 200$ and $k = 2484$ (yielding a total time span of 103.5 days).

17.4 Extraction of the $\mathrm{S_2}$ tide

Likewise, for $\mathbf{S_2}$, we have that

$$T_{\mathrm{S_2}} = 12.00 \text{ hr} \tag{339}$$

With $\Delta t = 1$ hr and 60 times the $\mathbf{S_2}$ period (so $M = 60$), we get

$$k = 60 \times 12.00 = 720 \tag{340}$$

(i.e., 720 hourly time steps).

17.5 Extraction of the $\mathbf{N_2}$ tide

For $\mathbf{N_2}$,

$$T_{\mathrm{N_2}} = 12.66 \text{ hr} \tag{341}$$

With $\Delta t = 1$ hr and 50 times the $\mathbf{N_2}$ period (so $M = 50$), we get

$$k = 50 \times 12.66 = 720 \tag{342}$$

(i.e., 720 hourly time steps).

17.6 Extraction of the $\mathbf{K_2}$ tide

For $\mathbf{K_2}$,

$$T_{\mathrm{K_2}} = 11.97 \text{ hr} \tag{343}$$

With $\Delta t = 1$ hr and 100 times the $\mathbf{K_2}$ period (so $M = 100$), we get

$$k = 100 \times 11.97 = 1197 \tag{344}$$

(i.e., 1197 hourly time steps).

17.7 Extraction of the $\mathbf{O_1}$ tide

For $\mathbf{O_1}$,

$$T_{\mathrm{O_1}} = 25.82 \text{ hr} \tag{345}$$

With $\Delta t = 1$ hr and 100 times the $\mathbf{O_1}$ period (so $M = 100$), we get

$$k = 100 \times 25.82 = 2582 \tag{346}$$

(i.e., 2582 hourly time steps).

17.8 Extraction of the $\mathbf{K_1}$ tide

For $\mathbf{K_1}$,

$$T_{\mathrm{K_1}} = 23.93 \text{ hr} \tag{347}$$

With $\Delta t = 1$ hr and 100 times the $\mathbf{K_1}$ period (so $M = 100$), we get

$$k = 100 \times 23.93 = 2393 \tag{348}$$

(i.e., 2393 hourly time steps).

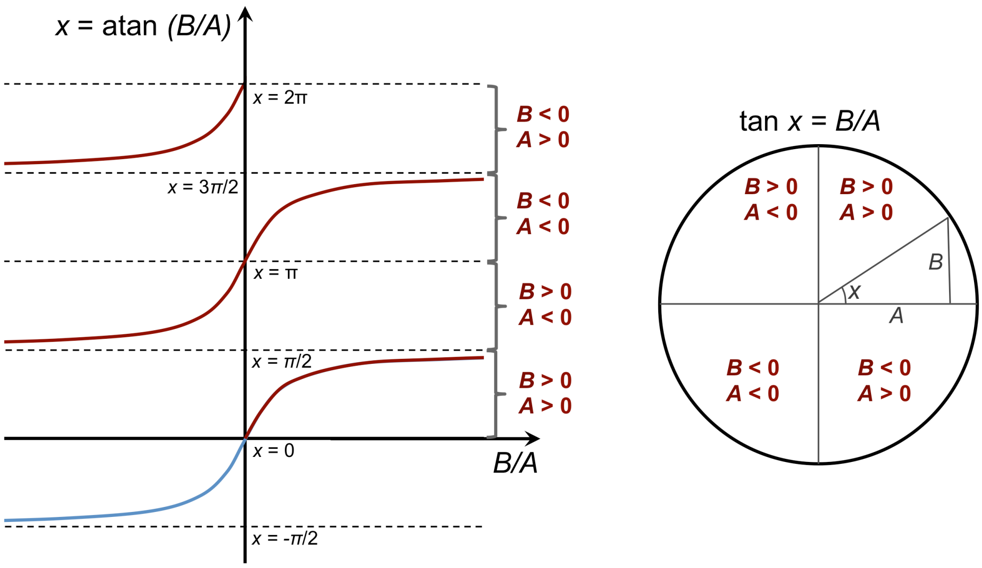

Figure 26: Right panel: Illustration of $\tan x = B/A$ on a unit circle, with the signs of the quantities A, B given (with $B > 0$ in the upper and $B < 0$ in the lower quadrants, and $A > 0$ in the right and $A < 0$ in the left quadrants, respectively). Left panel: Graph of $x = \arctan B/A$, with the signs of A, B consistent with the right panel.

17.9 Note 3

Care needs to be taken when computing g_m from expression (317), with values $0 \leq g_m < 2\pi$, since $\arctan x$ is defined for the interval $-\pi/2 < x < \pi/2$. $\arctan x$ can, however, be extended to the interval $0 \leq g_m < 2\pi$. The situation is outlined in Fig. 26.

Starting with $\tan x = B/A$, see right panel in Fig. 26, it follows that

$$0 \leq x < \pi/2 \quad \text{for} \quad B > 0, A > 0 \tag{349}$$

$$\pi/2 \leq x < \pi \quad \text{for} \quad B > 0, A < 0 \tag{350}$$

$$\pi \leq x < 3\pi/2 \quad \text{for} \quad B < 0, A < 0 \tag{351}$$

$$3\pi/2 \leq x < 2\pi \quad \text{for} \quad B < 0, A > 0 \tag{352}$$

Therefore, the four combinations of signs of A and B dictate which quadrant $x = \arctan B/A$ belongs to.

For any value of B_m/A_m, $\arctan B_m/A_m$ belongs to the open interval between $-\pi/2$ and $+\pi/2$, implying that g_m (see expression 317) is restricted to values between $-\pi/2$ and $\pi/2$.

However, based on the signs of A_m and B_m, we can uniquely construct the physically consistent value of g_m spanning out the entire interval from 0 to 2π. From the left panel in Fig. 26, the value of g_m can be computed according to the following algorithm:

$$B_m > 0, A_m > 0 : \qquad g_m = \arctan(B_m/A_m) \tag{353}$$

$$B_m > 0, A_m < 0 : \qquad g_m = \arctan(B_m/A_m) + \pi \tag{354}$$

$$B_m < 0, A_m < 0 : \qquad g_m = \arctan(B_m/A_m) + \pi \tag{355}$$

$$B_m < 0, A_m > 0 : \qquad g_m = \arctan(B_m/A_m) + 2\pi \tag{356}$$

Analysis tools like Python (see `https://www.w3schools.com/python/ref_math_atan2.asp`), Matlab (`https://www.mathworks.com/help/matlab/ref/atan2.html`) or R (`https://search.r-project.org/CRAN/refmans/raster/html/atan2.html`) all have embedded arctan functions that take into account the sign of A and B in $\arctan(B/A)$, commonly named atan2, called the '2-argument arctangent".

17.10 Note 4

- From the above, it follows that different lengths (or number of time steps) of the observed time series is used to extract the different tidal constituents. This is fine, but we need to ensure that the observed time series is sufficiently long so that all tidal constituents of interest are covered by the length of the observed time series.
- For the plotting, it is convenient to use a time interval covering a few fortnightly tides.
- For detailed inspection of the tidal constituents and the observed time series, 10 or so tidal periods may be plotted.
- There are several software packages for tidal analysis. Make e.g. a search `matlab tidal analysis` or `python tidal analysis` on the net.

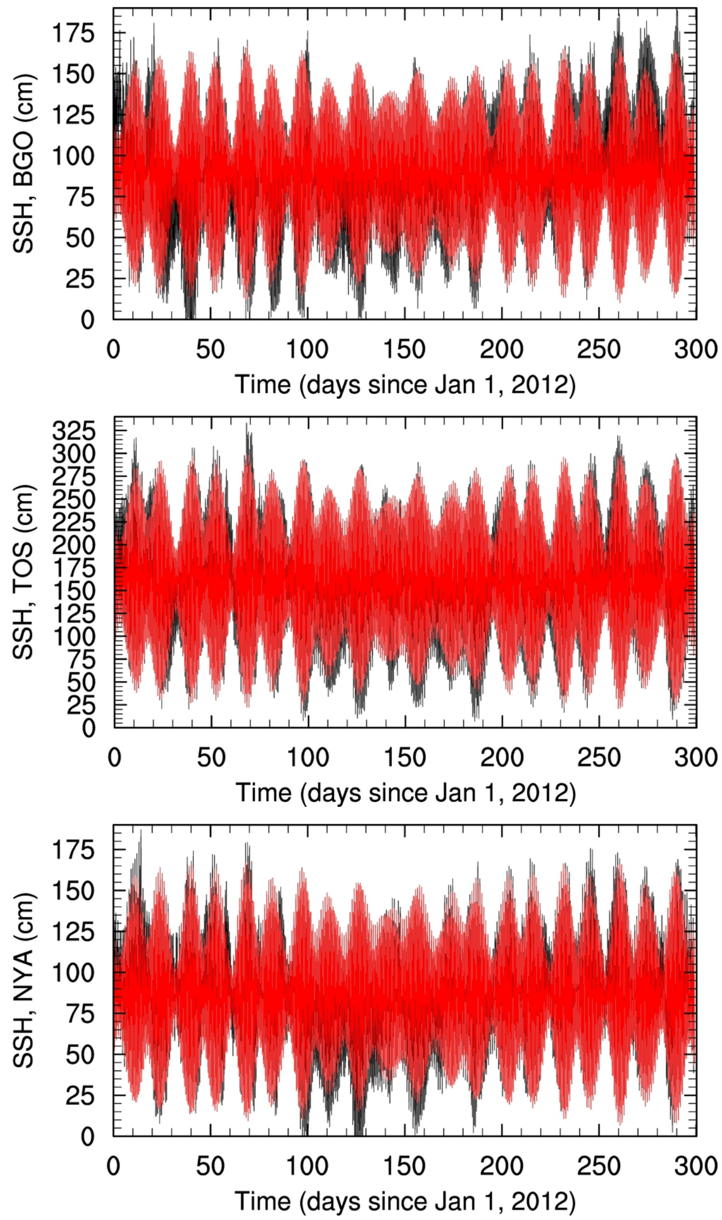

Figure 27: Observed (black line) and modelled (red line) sea surface hight (SSH, in m) at the Norwegian coastal cities Bergen (top panel), Tromsø (mid panel) and Ny-Ålesund (bottom panel) for the first 300 days of 2012. The modelled sea surface height is based on the algorithm described in the text, and includes the sum of the $\mathbf{M_2}$, $\mathbf{S_2}$, $\mathbf{N_2}$, $\mathbf{K_2}$, $\mathbf{O_1}$, and $\mathbf{K_1}$ constituents. The difference between the observed and modelled sea surface height can be attributed to an imperfect (simplified) model and the effect of weather. Note different ranges on the y-axes. Data from *Kartverket*, `https://www.kartverket.no/sehavniva/`.

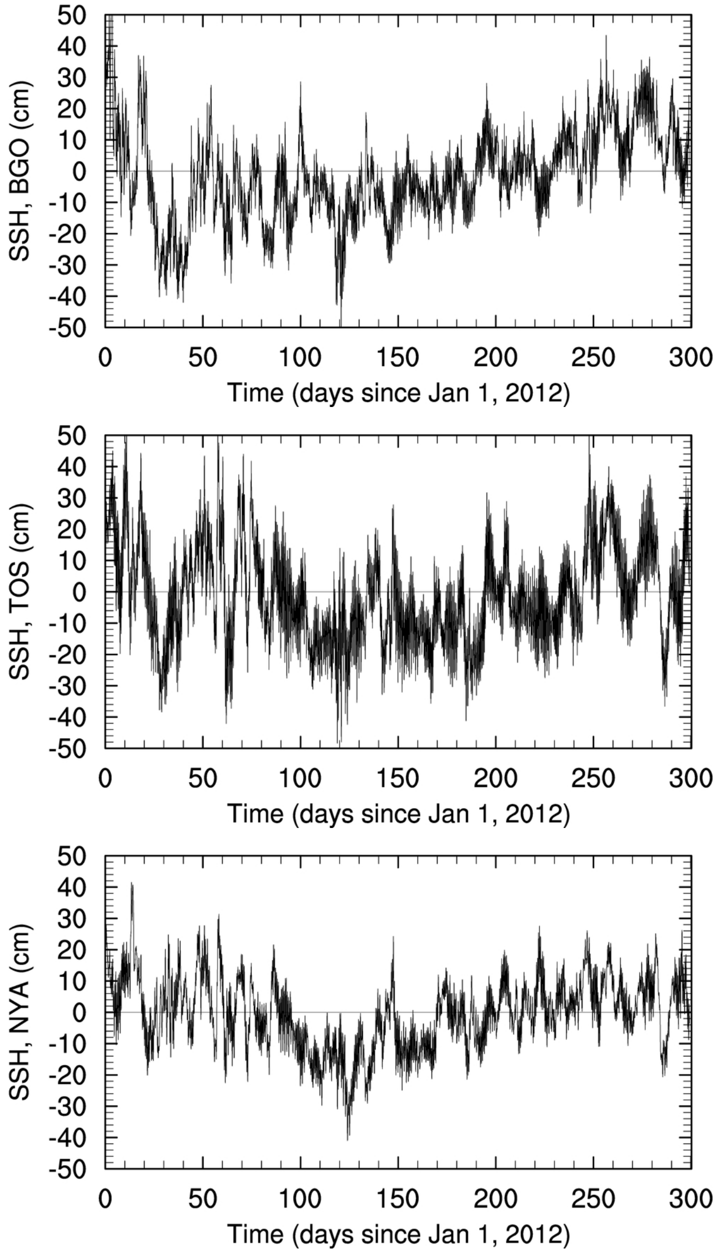

Figure 28: As Fig. 28, but showing the residuals, i.e., the difference between observed and modelled sea level height.

18 Tidal current analysis

As a starting point we assume that the tidal current components are on the form **/App. B/**

$$u = U\ \cos(\omega t - g_u) \tag{357}$$

$$v = V\ \cos(\omega t - g_v) \tag{358}$$

Here U, V are the current amplitudes, g_u, g_v (rad) are the phase angles, and ω (rad) is the speed of the tidal constituent under consideration. Note that for a given tidal constituent, the pair of unknowns U and g_u, as well as V and g_v, can be obtained similarly to the harmonic analysis of the sea surface height, see Sec. 17.

Based on expressions (357) and (358), characteristic properties of the tidal current, like the speed, the direction of flow, the semi-major and semi-minor axes, and the direction of the rotation can be derived.

18.1 Direction of flow θ and current speed q

From expressions (357) and (358), it follows directly that the direction of the flow θ is given by

$$\tan\theta = \frac{V\ \cos(\omega t - g_v)}{U\ \cos(\omega t - g_u)} \tag{359}$$

and that the current speed q, expressed in terms of $q^2 = u^2 + v^2$, equals

$$q^2 = U^2\ \cos^2(\omega t - g_u) + V^2\ \cos^2(\omega t - g_v) \tag{360}$$

18.2 Semi-major and semi-minor axes

By means of identity (629), it folllows that

$$q^2 = \frac{1}{2}\left\{U^2\left[1 + \cos\ 2(\omega t - g_u)\right] + V^2\left[1 + \cos\ 2(\omega t - g_v)\right]\right\} \tag{361}$$

$$= \frac{1}{2}\left\{U^2 + V^2 + U^2 \cos\ 2(\omega t - g_u) + V^2 \cos\ 2(\omega t - g_v)\right\} \tag{362}$$

The last cos-term the expression (362) can be expressed with the common factor $\omega t - g_u$, similarly to the derivation of the fortnightly signal (see expression 141):

$$\cos\ 2(\omega t - g_v) = \cos\ 2[\omega t - g_u + (g_u - g_v)] \tag{363}$$

$$\overset{(620)}{=} \cos\ 2(\omega t - g_u)\ \cos\ 2(g_u - g_v) - \sin\ 2(\omega t - g_u)\ \sin\ 2(g_u - g_v) \tag{364}$$

Thus,

$$q^2 = \frac{1}{2}(U^2 + V^2)\ + \frac{1}{2}\left[U^2 + V^2\ \cos\ 2(g_u - g_v)\right]\ \cos\ 2(\omega t - g_u) - \frac{1}{2}\,V^2\ \sin\ 2(g_u - g_v)\ \sin\ 2(\omega t - g_u) \tag{365}$$

Following the analysis of the fortnightly tide, see equations (146) and (147), an amplitude α and phase δ can be introduced by means of the expressions

$$U^2 + V^2\ \cos\ 2(g_u - g_v) = \alpha^2\ \cos\ 2\delta \tag{366}$$

$$V^2\ \sin\ 2(g_u - g_v) = \alpha^2\ \sin\ 2\delta \tag{367}$$

The sum of the square of the above equations determines α:

$$\begin{aligned} \alpha^4 &= [U^2 + V^2 \cos 2(g_u - g_v)]^2 + [V^2 \sin 2(g_u - g_v)]^2 \\ &= U^4 + V^4 + 2\,U^2V^2 \cos 2(g_u - g_v) \end{aligned} \tag{368}$$

Furthermore, expression (367) divided by (366) determines δ:

$$\tan 2\delta = \frac{V^2 \sin 2(g_u - g_v)}{U^2 + V^2 \cos 2(g_u - g_v)} \tag{369}$$

The introduction of α and δ implies that expression (365) can be written as

$$\begin{aligned} q^2 &= \frac{1}{2}(U^2 + V^2) + \frac{1}{2}\,\alpha^2 \cos 2\delta \cos 2(\omega t - g_u) - \frac{1}{2}\,\alpha^2 \sin 2\delta \sin 2(\omega t - g_u) \\ &= \frac{1}{2}\,[U^2 + V^2] + \frac{1}{2}\,\alpha^2 \cos 2(\omega t - g_u + \delta) \\ &\overset{(630)}{=} \frac{1}{2}\,[U^2 + V^2 - \alpha^2] + \alpha^2 \cos^2(\omega t - g_u + \delta) \end{aligned} \tag{370}$$

Consequently, the maximum current speed $q_{\max}$, or the semi-major current axis, is given by

$$q_{\max}^2 = \frac{1}{2}\,[U^2 + V^2 + \alpha^2] \tag{371}$$

Likewise, the minimum current speed $q_{\min}$, or the semi-minor current axis, is

$$q_{\min}^2 = \frac{1}{2}\,[U^2 + V^2 - \alpha^2] \tag{372}$$

18.3 Time of maximum current speed

From expression (370), the maximum current speed occurs when

$$\omega t - g_u + \delta = m\,\pi\,, \qquad \text{where} \qquad m = 0, 1, 2, ... \tag{373}$$

18.4 Direction of maximum current speed

Expression (373) with, for simplicity, $m = 0$, gives the following relationship for the cosine-argument in (370) at the time of maximum speed

$$\omega t - g_u = -\delta \tag{374}$$

The direction of the maximum current speed follows then directly from expression (359),

$$\theta_{\max} = \arctan \frac{V \cos(g_u - g_v - \delta)}{U \cos\delta} \tag{375}$$

18.5 Direction of rotation

The direction of the rotation is given by the sign of $d\theta/dt$. To simplify the differentiation, expression (359) can be put in the form

$$\tan\theta = \frac{V \cos(\omega t - g_v)}{U \cos(\omega t - g_u)} = x \tag{376}$$

where x has been introduced for convenience. We than have that

$$\theta = \arctan x \tag{377}$$

The change of θ with time is then given by the chain rule

$$\frac{\partial \theta}{\partial t} = \frac{\partial \theta}{\partial x}\frac{\partial x}{\partial t} \overset{(643)}{=} \frac{1}{1+x^2}\frac{\partial x}{\partial t} \tag{378}$$

The sign of $\partial\theta/\partial t$ is therefore given by the sign of $\partial x/\partial t$. Upon differentiation,

$$\frac{\partial x}{\partial t} = \frac{UV\omega}{U^2\ \cos^2(\omega t - g_u)}\ \sin(g_v - g_u) \tag{379}$$

Since the above fraction is always positive, the direction of rotation, $\partial\theta/\partial t$, is governed by the difference $g_v - g_u$:

$$0 < g_v - g_u < \pi \quad \text{gives} \quad \frac{\partial \theta}{\partial t} > 0\,, \quad \text{or anticlockwise rotation,} \tag{380}$$

$$\pi < g_v - g_u < 2\pi \quad \text{gives} \quad \frac{\partial \theta}{\partial t} < 0\,, \quad \text{or clockwise rotation,} \tag{381}$$

$$g_v - g_u = 0, \pi, 2\pi, ... \quad \text{gives} \quad \frac{\partial \theta}{\partial t} = 0\,, \quad \text{or rectilinear flow.} \tag{382}$$

Side note for the sake of completeness: From the definition of x given by expression (376), we get

$$1 + x^2 = \frac{q^2}{U^2\ \cos^2(\omega t - g_u)}, \qquad \text{so} \qquad \frac{1}{1+x^2} = \frac{U^2\ \cos^2(\omega t - g_u)}{q^2} \tag{383}$$

With $\partial x/\partial t$ given by (379), the following expression gives the temporal derivative of θ:

$$\frac{\partial \theta}{\partial t} = \frac{UV\omega}{q^2}\ \sin(g_v - g_u) \tag{384}$$

As stated above, the sign of the above expression is identical to that of expression (379), i.e., the sign is given by the phase angle difference $g_v - g_u$.

Part IV

Tidal related wave dynamics

19 The shallow water equations

A wave is a physical process that transport information – such as surface elevation or energy – in time and space, without or with little advection of mass associated with the transport. This is in contrast to the geostrophic balance and ageostrophic flow in the atmosphere and ocean that is always associated with advection of mass. Accordingly, a wave signal is propagated without or with little influence of the Earth's rotation, although the propagation velocity can be large. In contrast, advection of mass will always be influenced by the Coriolis effect, linearly increasing with the speed of the fluid.

A general overview of definitions and key properties of waves is given in the appendix.

19.1 Starting point and configuration

The basis for the shallow water equations is the standard form of the horizontal momentum equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \mathcal{F}_x \tag{385}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \mathcal{F}_y \tag{386}$$

and the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\, \mathbf{u}) = 0 \tag{387}$$

with $\mathbf{u} = (u, v)$.

We consider a homogeneous ($\rho = \rho_0$), frictionless ($\mathcal{F}_H = 0$) and barotropic ($\partial u/\partial z = \partial v/\partial z = 0$) fluid with free surface $\zeta(x, y, t)$ as illustrated in Fig. 29. The equations (385)–(387) can then be expressed as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \tag{388}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \tag{389}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{390}$$

19.2 The continuity equation

In the che continuity equation (390), u and v are independent of z (due to the assumption a barotropic fluid), but $\partial u/\partial x \neq 0$ and $\partial v/\partial y \neq 0$, so that we can have divergent flow.

We consider the homogeneous fluid shown in Fig. 29 and integrates (390) from the bottom $z = b(x, y)$ to the free surface $z = b(x, y) + h(x, y, t)$. This gives

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\int_b^{b+h} dz + w\big|_b^{b+h} = 0 \tag{391}$$

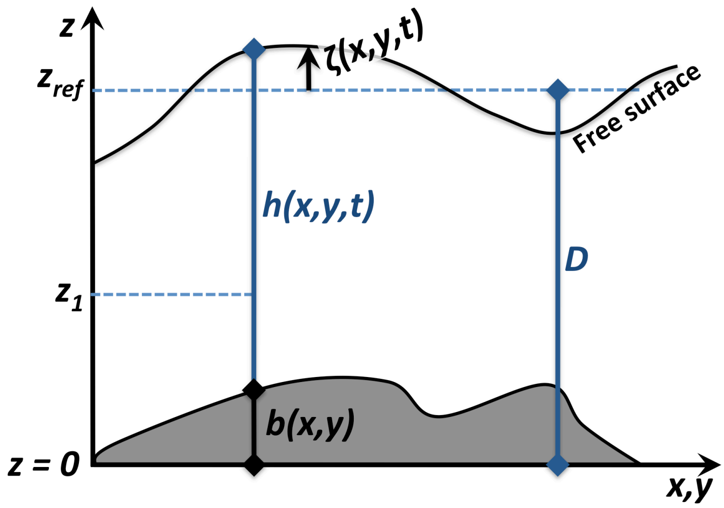

Figure 29: Illustration of a homogeneous væ fluid with free surface $\zeta(x, y, t)$ and general bathymetry $b(x, y)$. $h(x, y, t)$ describes the total depth of the fluid. $z_{\text{ref}} = D$ is a reference level which describes the mean surface height.

In the above expression, $w(z = b + h)$ is the motion of the free surface. This can be expressed using the total derivative (describing how the surface changes *with* the motion)

$$\begin{aligned} w(z = b + h) &= \left.\frac{Dz}{dt}\right|_{b+h} = \frac{D}{dt}(b + h) \\ &= \frac{\partial}{\partial t}(b + h) + u\frac{\partial}{\partial x}(b + h) + v\frac{\partial}{\partial y}(b + h) \\ &= \frac{\partial h}{\partial t} + u\frac{\partial}{\partial x}(b + h) + v\frac{\partial}{\partial y}(b + h) \end{aligned} \tag{392}$$

Similarily,

$$w(z = b) = \frac{db}{dt} = u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y} \tag{393}$$

Expressions (392) and (393) inserted in (391) gives

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(b + hb) + \frac{\partial h}{\partial t} + u\frac{\partial}{\partial x}(b + h) + v\frac{\partial}{\partial y}(b + h) - u\frac{\partial b}{\partial x} - v\frac{\partial b}{\partial y} = 0 \tag{394}$$

or

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h\,u) + \frac{\partial}{\partial y}(h\,v) = 0 \tag{395}$$

which is the continuity equation expressed by the local change of the fluid thickness h and divergence of $h\mathbf{u}_H$.

Alternatively, since

$$h(x, y, t) + b(x, y) = D + \zeta(x, y, t) \tag{396}$$

the time derivative in (395) can be is expressed by ζ

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0 \tag{397}$$

19.3 The momentum equations

The horizontal momentum equations is given by (388) and (389).

The pressure $p(x, y, z, t)$ is given by the hydrostatic equation

$$\frac{\partial p}{\partial z} = -g\rho_0 \tag{398}$$

Pressure fluctuations due to changes to the surface height can be expressed as

$$p|_D^{b+h} = -g\rho_0(b + h - D) \tag{399}$$

(399) can be written

$$p_s - p(D) = -g\rho_0\zeta \tag{400}$$

or

$$p(D) = p_s + g\rho_0\zeta(x, y, t) \tag{401}$$

Similarly, for an arbitrarily depth z_1 (see Fig. 29),

$$p(z_1) = g\rho_0\Delta z + p_s + g\rho_0\zeta(x, y, t) \tag{402}$$

where $\Delta z = D - z_1$.

Over time, there are (very) small spatial variations in the surface pressure p_s. We can therefore ignore the contribution from ∇p_s. From the expressions (401) and (402) it then follows that

$$\frac{\partial p(D)}{\partial x} = \frac{\partial p(z_1)}{\partial x} = g\rho_0\frac{\partial \zeta}{\partial x} \tag{403}$$

The same relationship holds for $\partial/\partial y$. It is therefore only the surface displacement ζ which gives rise to the pressure force in a homogeneous fluid. For this reason $p = g\rho_0\zeta$ is called the *dynamic pressure.*

19.4 The final set of equations

The above gives the shallow water equations expressed in terms of u, v, h and ζ

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial \zeta}{\partial x} \tag{404}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial \zeta}{\partial y} \tag{405}$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0 \tag{406}$$

For flat bottom we have that

$$h(x, y, t) = D + \zeta(x, y, t) \tag{407}$$

Note that the phrase *shallow water equations* do not imply that the waves are found in shallow waters. Rather, the phrase implies that the wavelength of the waves are much longer than the thickness of the fluid.

20 Gravity waves

20.1 One-dimensional gravity waves

Gravity waves occur at the boundary between the ocean and atmosphere, or more generally between two (or more) fluids with different densities. In the following we consider surface waves. The force acting on these waves is gravity, hence the name (surface) gravity waves.

If the waves describe small fluctuations in the wave variables, the wave equation (404)–(406) can be linearized. This means that any product of wave variables, such as the advection terms in (404) and (405), can be neglected. Furthermore, we start by neglecting the effect of Earth's rotation, implying that the local derivative term ($\partial/\partial t$) is much larger than the Coriolis-term (the effect of the Earth's rotation is discussed in the following section).

With $f = 0$ and by considering a one-dimensional wave movement in the x-direction ($v = 0$ and $\partial/\partial y = 0$), equation (404) and (406) can be written as **/5.3/**

$$\frac{\partial u}{\partial t} = -g\frac{\partial \zeta}{\partial x} \quad (408)$$

$$\frac{\partial \zeta}{\partial t} = -D\frac{\partial u}{\partial x} \quad (409)$$

The velocity component u can be eliminated from the above expressions by considering $\partial(408)/\partial x$ and $\partial(409)\partial t$, giving the classical wave equation

$$\frac{\partial^2 \zeta}{\partial t^2} - gD\frac{\partial^2 \zeta}{\partial x^2} = 0 \quad (410)$$

The wave equation (410) can be solved by seeking solutions of the form (see appendix F)

$$\zeta = \mathrm{Re}\left\{\overline{\zeta}_0 \exp[i(kx - \omega t)]\right\} \quad (411)$$

where Re denotes the real part, $\overline{\zeta}_0$ is the amplitude, k is the wave number in the x-direction and ω is the angular frequency.

Insertion of (411) into (410) leads to the dispersion relation

$$-\omega^2 + gD\,k^2 = 0 \quad (412)$$

Since the wave's phase velocity c is given by $c = \omega/k$ (see appendix F), the phase velocity of the gravity wave c is given by **/p. 100/**

$$c = c_0 = \pm\sqrt{gD} \quad (413)$$

The phase velocity c_0 is independent of the wave number k, so the gravity wave is *non-dispersive*. Gravity waves in a homogeneous, smooth and barotrop fluid propagate therefore with a phase velocity proportional to $\sqrt{gD}$, and are independent of the wave wave number or wavelength.

20.1.1 Typical wavelengths for tidal waves

Since $\omega = 2\pi/T$ and $k = 2\pi/\lambda$, it follows that

$$c = \frac{\omega}{k} = \frac{\lambda}{T} = \sqrt{gD} \quad (414)$$

where the last equality is given by (413). Consequently,

$$\lambda = T\sqrt{gD} \tag{415}$$

For the diurnal and semi-diurnal tides, the periods are $T_1 \approx 24$ hr and $T_2 \approx 12$ hr, respectively. The resulting wavelengths for some ocean depths D are given in table 8.

Depth D (m)	Period, diurnal tide $T_1 \approx 24$ hr	Period, semi-diurnal tide $T_2 \approx 12$ hr
20	$\lambda_1 \approx 1200$	$\lambda_2 \approx 600$
50	1900	950
100	8600	4300

Table 8: Typical wavelengths λ_1 (km) and λ_2 (km) for diurnal and semi-durnal tides with approximate periods T_1 and T_2, and for three ocean depths D (m).

It follows from the table that the gravity waves – and by that waves governed by tidal forces – have very long wavelengths. Only at very shallow water, at depths less than 20 m for diurnal tides and less than 50 m for semi-diurnal tides, the wavelengths are less than 1000 km. The tidal wavelengths are therefore, in general, much longer than the depth of the ocean, in accordance with the phrase *shallow water equations* as described by the end of Sec. 19.4. Note that in shallow water, friction needs to be included, so the above comment is an oversimplification. The general finding, that tidal waves have long wave lengths, is still valid.

20.2 One-dimensional gravity waves in a closed channel

An externally forced ocean basin or ocean channel will lead to standing waves with wavelengths determined by the geometry of the basin/channel. The resulting standing waves are the *eigenmodes* of the system, also called *seiches*.

For a closed channel with length L (m) and with boundaries at $x = 0$ and $x = L$, kinematic boundary conditions impose no flow across the boundary: $u|_{x=0} = u|_{x=L} = 0$. From equation (408), this results in the two boundary conditions:

$$\left.\frac{\partial \zeta}{\partial x}\right|_{x=0} = \left.\frac{\partial \zeta}{\partial x}\right|_{x=L} = 0 \tag{416}$$

If we assume that the surface elevation ζ can be expressed by means of trigonometric waves travelling in both the positive and negative x-direction, the surface elevation can be written as

$$\zeta = A\exp[i(\omega t - kx)] + B\exp[i(\omega t + kx)] \tag{417}$$

where A and B are the amplitudes for waves travelling in the positive and negative x-directions, respectively. Thus,

$$\frac{\partial \zeta}{\partial x} = -ikA\exp[i(\omega t - kx)] + ikB\exp[i(\omega t + kx)] \tag{418}$$

which, by means of the boundary condition at $x = 0$ in expression (416) yields $A = B$. Therefore,

$$\begin{aligned} \zeta &= A\exp[i(\omega t - kx)] + \exp[i(\omega t + kx)] \\ &= A\{\exp(-ikx) + \exp(ikx)\}\exp(i\omega t) \\ &= 2A\cos(kx)\exp(i\omega t) \end{aligned} \tag{419}$$

where the identity

$$\exp(\pm i\varphi) = \cos\varphi \pm i\sin\varphi \tag{420}$$

is used in the last equality.

At the other end of the cannel, at $x = L$, we get that

$$\left.\frac{\partial\zeta}{\partial x}\right|_{x=L} = \left.-2kA\sin(kx)\exp(i\omega t)\right|_{x=L} = 0 \tag{421}$$

which is satisfied for

$$kL = n\pi\,, \qquad n = 1, 2, ... \tag{422}$$

Note that $n = 0$ is also a solution of the above, but in this case $k = 0$, implying infinitely long wavelengths. Such waves do not exist on a finite ocean, limiting $n = 1, 2, \ldots$.

From expression (414), it follows that

$$k = \frac{\omega}{\sqrt{gD}} = \frac{2\pi}{T\sqrt{gD}} \tag{423}$$

where $\omega = 2\pi/T$ is used in the last equality. By combining (422) and (423), the following eigenmode, natural or seiche periods are obtained: **/5.6 & 7.8/**

$$\boxed{T_n = \frac{2L}{n\sqrt{gD}}\,, \qquad n = 1, 2, 3, ...} \tag{424}$$

The longest period T_1 occurs for $n = 1$, with the following periods equal to $1/2$, $1/3$, $1/4$,... of T_1. $T_1 = 2L/\sqrt{gD}$ is known as *Merian's formula*.

Since $T_1 = 2L/\sqrt{gD}$ and $c = \sqrt{gD}$, we get that

$$T_1 = \frac{2L}{c} \tag{425}$$

In addition, by using the definitions $\lambda = 2\pi/k$ and $c = \omega/k$, we have that

$$T_1 = \frac{2\pi}{\omega} = \frac{2\pi}{c\,k} = \frac{2\pi}{c}\frac{\lambda}{2\pi} = \frac{\lambda}{c} \tag{426}$$

From (425) and (426), it follows that

$$\lambda = 2L \tag{427}$$

Thus, for T_1, the wavelength is twice the length of the channel, implying that we have two equal waves, travelling in opposite directions, yielding a standing wave pattern.

20.3 One-dimensional gravity waves in a semi-closed channel

If the channel is open in one end, in our case at $x = L$, the infinite reservoir of water outside the channel dictates the surface elevation at the opening of the channel. In this case expression (419) at $x = L$ becomes

$$\zeta|_{x=L} = 2A\cos(kx)\exp(i\omega t)|_{x=L} = A'\exp(i\omega' t) \tag{428}$$

where A' and ω' are the amplitude and the frequency of the externally forced, large-scale ocean tide. Consequently,

$$2A = A'\frac{\exp(i\omega' t)}{\cos(kL)\exp(i\omega t)} \tag{429}$$

and from equation (419),

$$\zeta = A'\exp(i\omega' t)\frac{\cos(kx)}{\cos(kL)} \tag{430}$$

The channel response is largest, leading to resonance, when $\cos(kL) \to 0$, or for

$$kL = n\frac{\pi}{2}\,, \qquad n = 1, 3, 5, ... \tag{431}$$

Expression (423) and (431) give **/7.9/**

$$\boxed{T_n = \frac{4L}{n\sqrt{gD}}\,, \qquad n = 1, 3, 5, ...} \tag{432}$$

20.3.1 Resonance

Any combinations of D and L in expression (432) yielding periods around 12 or 24 hours will lead to resonance with the semi-diurnal or diurnal tides, giving rise to particularly high tides at the end (the head) of the bay or channel.

Should, for instance, D be in the range 30–35 m and $L = 200$ km, T_1 is around 12 hr and is thus close to the period of the semi-diurnal tides, resulting in particularly high tides at the head of the bay or channel. The given depth and length scales are representative for the Severn Estuary between England and Wales, explaining the large tidal variations of 10–15 m in the region.

For periods close to 12 hr (as an example), larger values of n, yield resonance for smaller values of $L/\sqrt{D}$. So resonance can be found in fjords, bays and channels with different configurations.

21 Sverdrup waves and other related waves

Also here we consider long surface waves, implying waves with wavelengths much longer than the thickness of the fluid. The effect of Earth's rotation is now considered by introducing the Coriolis-terms in the momentum equations. Variations in the Earth's rotation are, however, neglected, so $f = \text{const.}$

Moreover, it is assumed that the waves has infinite horizontal extent in the two horizontal dimensions. The resulting waves are often called the Sverdrup waves. The basic equations follows from (404)–(406)

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \zeta}{\partial x} \quad (433)$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \zeta}{\partial y} \quad (434)$$

$$\frac{\partial \zeta}{\partial t} + D\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \quad (435)$$

21.1 Resulting dispersion relationship

If we assume a solution on the form

$$u, v, \zeta \propto \exp[i\,(kx + ly - \omega t)] \quad (436)$$

where $k = 2\pi/\lambda_x$ and $l = 2\pi/\lambda_y$ are the wave numbers in the x- and y-directions (and $\lambda_{x,y}$ are the corresponding wavelengths, see Sec. F), respectively, the following algebraic relations are obtained

$$-i\omega u - fv = -igk\zeta \quad (437)$$

$$-i\omega v + fu = -igl\zeta \quad (438)$$

$$-i\omega\zeta + D(ku + lv) = 0 \quad (439)$$

The equations (437)–(439) constitute a set of three equations with three unknowns, and can be expressed in matrix form as

$$\begin{pmatrix} -i\omega & -f & igk \\ f & -i\omega & igl \\ iDk & iDl & -i\omega \end{pmatrix} \begin{pmatrix} u \\ v \\ \zeta \end{pmatrix} = \mathbf{0} \quad (440)$$

Non-trivial solutions exist when the equation system's determinant vanishes. This gives the following relationship

$$\omega[\omega^2 - f^2 - gD(k^2 + l^2)] = 0 \quad (441)$$

or, by introducing the horizontal wave number $k_h^2 = k^2 + l^2$,

$$\omega\,[\omega^2 - f^2 - gDk_h^2] = 0 \quad (442)$$

If the phase speed of the gravity waves $c_0^2 = gD$ is introduced, one obtains

$$\omega[\omega^2 - f^2 - c_0^2\,k_h^2] = 0 \quad (443)$$

The expressions (442) and (443) are the *dispersion relationship* of the Sverdrup waves. It is generally different physical mechanisms involved for the different solutions of the dispersion relationship. It is therefore convenient to discuss the different solutions separately.

Since the expression in square brackets in (442) and (443) includes the wave number k_h, the Sverdrup waves are *dispersive*. Waves with different wave numbers (or wavelengths) will therefore propagate at different speeds.

21.1.1 Symmetry

The main part of the dispersion relationship, the one within the square brackets in expression (443), is symmetrical with respect to the x- and y-directions. This means that neither the x- or y-direction has a special significance for the wave field. We can therefore orient the coordinate system such that the x-axis is aligned with the direction of the wave propagation. In this case, $k_h = k$ and $l = 0$. The resulting dispersion relationship then becomes

$$\omega\,[\omega^2 - f^2 - c_0^2\,k^2] = 0 \tag{444}$$

The above expression has two, physically different solutions,

$$\omega = 0 \tag{445}$$

and

$$\omega^2 = f^2 + c_0^2\,k^2 \tag{446}$$

21.2 Case $\omega = 0$

The trivial solution $\omega = 0$ from (446) is consistent with $\partial/\partial t = 0$. From the governing equations (433) and (434), we see that this solution is the geostrophic force balance. The stationary (i.e., time-invariant) solution of the shallow water equations is thus geostrophic balance.

21.3 Case $\omega^2 = f^2 + c_0^2\,k^2$

For

$$\omega^2 = f^2 + c_0^2\,k^2 \tag{447}$$

$|\omega| \geq |f|$ for all possible wave solutions.

The phase speed c is given by $c = \omega/k$. Thus,

$$\begin{aligned} c = \frac{\omega}{k} &= \frac{1}{k}\left(f^2 + c_0^2\,k^2\right)^{1/2} \\ &= c_0\left(\frac{f^2}{k^2\,c_0^2} + 1\right)^{1/2} \\ &= c_0\left(1 + \frac{1}{k^2\,L_\rho^2}\right)^{1/2} \end{aligned} \tag{448}$$

where $L_\rho = c_0/f$ is the *Rossby deformation radius*. The above equation shows that the phase speed c increases with decreasing wave number or increasing wavelength.

The group velocity of the wave is given by $c_g = \partial\omega/\partial k$. Differentiation of the dispersion relationship (446) gives

$$2\omega\,\partial\omega = 2kc_0^2\,\partial k \tag{449}$$

or

$$c_g = \frac{k}{\omega}\,c_0^2 = \frac{c_0^2}{c} \tag{450}$$

Accordingly, the product $c\,c_g$ equals the constant c_0^2. The group speed therefore decreases with increasing wavelengths.

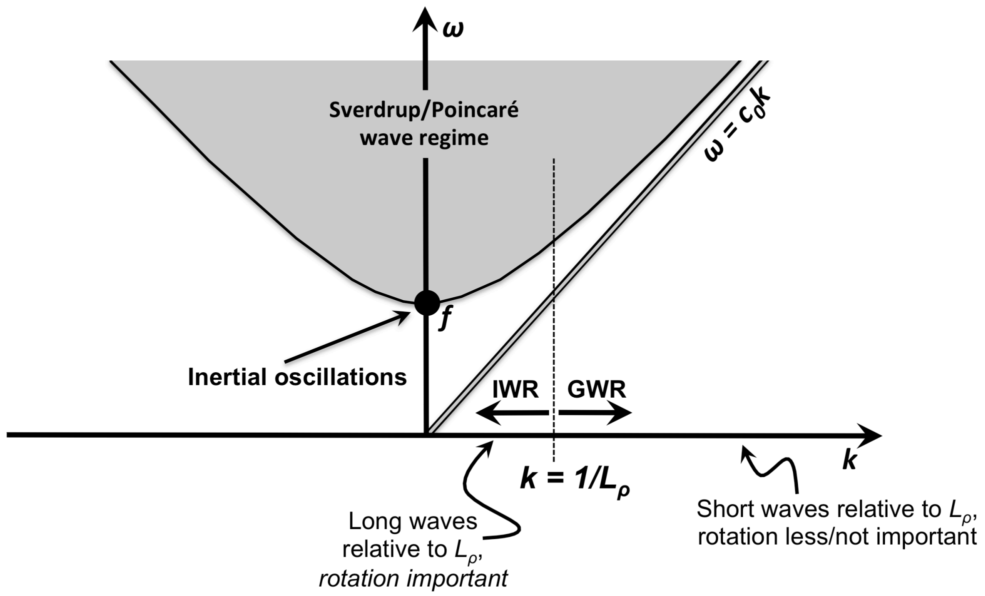

Figure 30: Graphic representation of the dispersion relation given by (446). In addition, the dispersion relation for gravity waves (Sec. 20.1) and inertial oscillations (Sec. 21.6) are shown. L_ρ is the Rossby deformation radius, IWR is the inertial wave regime where the Earth's rotation is important, and GWR is the gravity wave regime where the effect of Earth's rotation is of small or vanishing importance.

All possible wave solutions given by (446), expressed in terms of the wave number k and frequency ω, are illustrated in Fig. 30. For small wave number k (long waves), the wave properties follows those of the near inertia waves (see Sec. 21.6) and the effect of the Earth's rotation f is important. For large wave numbers (short waves), the effect of Earth's rotation has little effect. In the latter case, the wave regime is associated with gravitational waves (Sec. 20.1).

21.3.1 The resulting wave motion

For $l = 0$, it follows from (437) and (438) that

$$-i\omega u - fv = -igk\bar{\zeta}_0 \tag{451}$$

and

$$-i\omega v + fu = 0 \tag{452}$$

From (452),

$$u = i\,\frac{\omega}{f}v \tag{453}$$

which inserted in (451) gives

$$\frac{\omega^2 - f^2}{f}v = -igk\zeta \tag{454}$$

Since, from expression (446),

$$\omega^2 - f^2 = gDk^2 \tag{455}$$

one gets that

$$v = -i\frac{f}{kD}\zeta \tag{456}$$

so u from (453) becomes

$$u = \frac{\omega}{kD}\zeta \tag{457}$$

Since $|\omega| \geq |f|$, it follows from (456) and (457) that $|u| \geq |v|$, or that the wave has a net propagation in the x-direction.

Only the real part of the solution has a physical meaning. If we assume that the amplitude $\overline{\zeta}_0$ is real, it follows that

$$\zeta = \mathrm{Re}\{\overline{\zeta}_0 \exp[i(kx - \omega t)]\} = \overline{\zeta}_0 \cos(kx - \omega t) \tag{458}$$

(458) inserted in (457) and (456) gives

$$u = \omega\,\frac{\overline{\zeta}_0}{kD}\,\cos(kx - \omega t) \tag{459}$$

$$v = f\,\frac{\overline{\zeta}_0}{kD}\,\sin(kx - \omega t) \tag{460}$$

Thus, the Earth's rotation is not felt in the x-direction, while the y-direction is influenced by f.

For a fixed point in space, for example at $x = 0$, the following temporal change takes place, assuming $f > 0$ (northern hemisphere) and by using $\overline{\zeta}_0' = \overline{\zeta}_0/(kD) > 0$:

$$\omega t = 0: \qquad u = \omega\overline{\zeta}_0' > 0,\ v = 0 \tag{461}$$

$$\omega t = \pi/2: \qquad u = 0,\ v = -f\overline{\zeta}_0' < 0 \tag{462}$$

$$\omega t = \pi: \qquad u = -\omega\overline{\zeta}_0' < 0,\ v = 0 \tag{463}$$

$$\omega t = 3\pi/2: \qquad u = 0,\ v = f\overline{\zeta}_0' > 0 \tag{464}$$

The speed components span out ellipses as shown in the upper panel of Fig. 31 with a resultant movement directed clockwise in the Northern Hemisphere. Moreover, the ellipse's major axis is directed in the x -direction, i.e., in the direction of the wave propagation. The ratio between the semi-major and semi-minor axes is given by $|\omega|/|f|$.

A particle's position is found by integrating (459) and (460) with respect to time t. This gives

$$x \propto -\sin(-\omega t) \tag{465}$$

$$y \propto \cos(-\omega t) \tag{466}$$

The resulting motion is directed *with* the clock (anti-cyclonic rotation) in the northern hemisphere, see bottom panel of Fig. 31. Due to variations in water depth, the coastline geometry and non-linear wave-wave interactions, the horizontal tidal movement can be both cyclonic and anti-cyclonic. But for an infinite ocean with flat bottom, which are the underlying assumptions for the Sverdrup waves, the tidal ellipses describe anti-cyclonic movement.

The spatial characteristics at a given time, for example at $t = 0$, yields the relationships shown in Table (9).

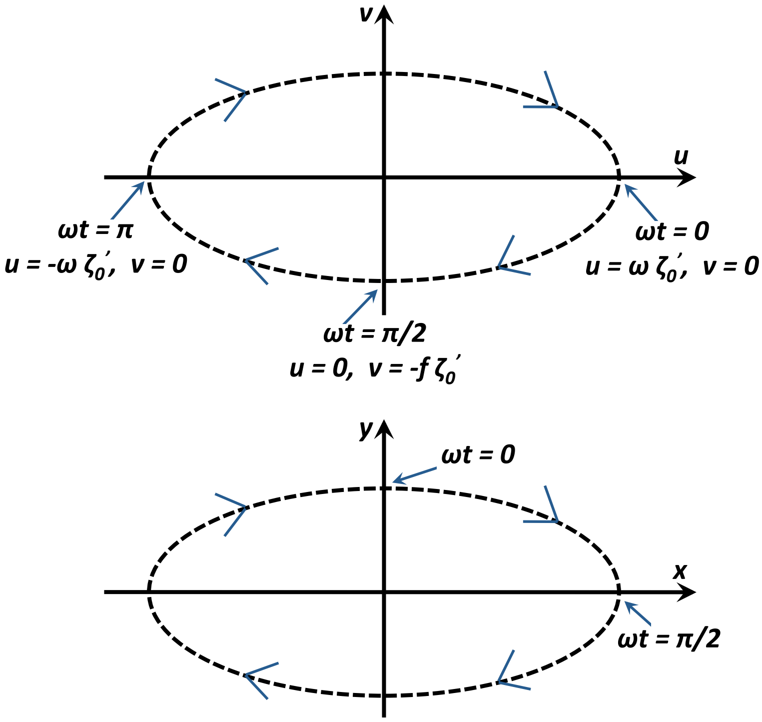

Figure 31: Illustration of the temporal change of the u- and v-components (top panel) and the corresponding x- and y-positions (lower panel) in the northern hemisphere. The resulting motion is directed *with* the clock (anti-cyclonic rotation).

Variable:	x	u	y	v	ζ
Amplitude:	$\omega\overline{\zeta}_0'/k$	$\omega\overline{\zeta}_0'$	$f\overline{\zeta}_0'/k$	$f\overline{\zeta}_0'$	$\overline{\zeta}_0$
	Sign:				
$kx = 0$	0	$+$	$-$	0	$+$
$kx = \pi/2$	$+$	0	0	$+$	0
$kx = \pi$	0	$-$	$+$	0	$-$
$kx = 3\pi/2$	$-$	0	0	$-$	0

Table 9: Overview of the magnitude and sign of x, y, u, v and ζ for four values of the argument kx, and for $t = 0$.

The corresponding movement in the xy-plane is shown in the lower part of Fig. (32). In this figure, the velocity-component in the direction of the wave propagation, u, is shown with red horizontal arrows. It follows that there is convergence in the left half and divergence in the right half of the figure. Therefore, the surface is lifted in the convergent zone and lowered where divergence takes place (red vertical arrows in the figure). Since the wave form is fixed, the wave propagates in the positive x-direction. It is therefore the fluid oscillation in the x-direction, resulting in convergence and divergence in the fluid, that drives the wave forward.

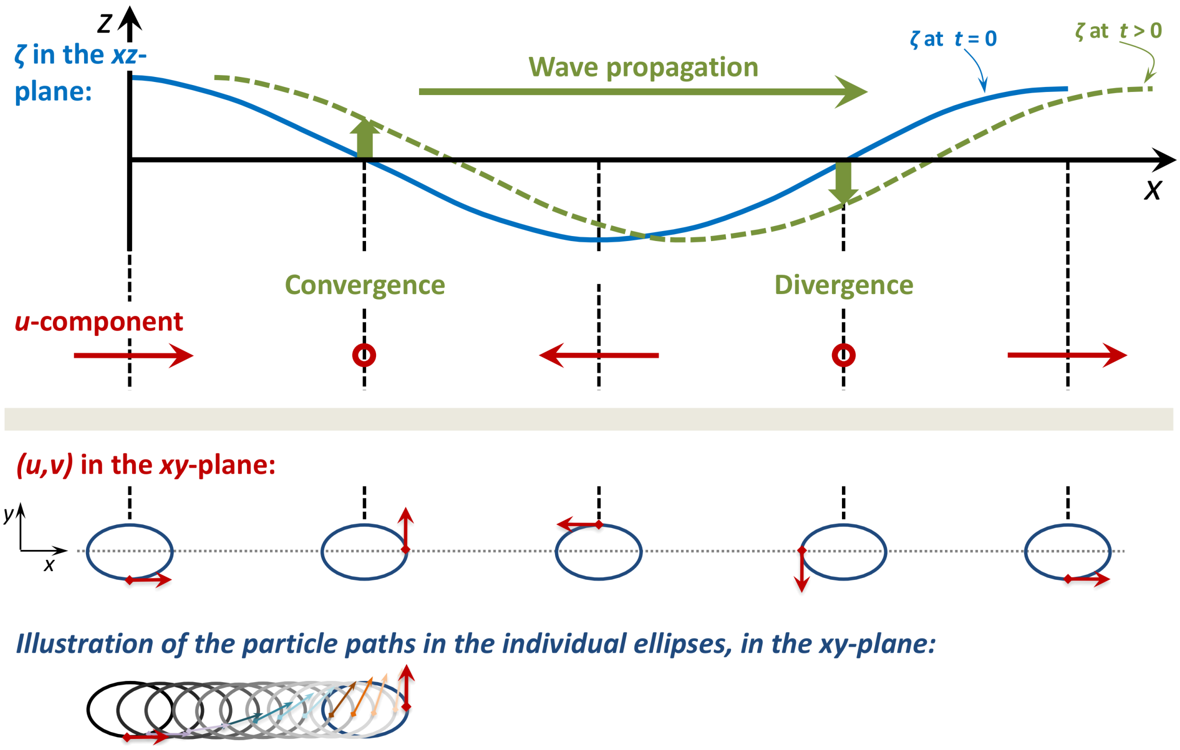

Figure 32: Illustration of the surface elevation ζ and the u-component of fluid particles in the vertical xz-plane at $t = 0$ or for any constant time (upper part of the figure), and the corresponding particle movements in the horizontal xy-plane (ellipses in the mid part of the figure), representing northern hemisphere Sverdrup waves. “Convergence” and “Divergence” denote, respectively, convergent and divergent flow in the x-direction. The current vector for the (many) individual ellipses between $kx = 0$ and $kx = \pi/2$ are sketched at the bottom of the figure.

21.3.2 Approximate length of the tidal axes

If the horizontal tidal component in the direction of the propagation of the tidal wave is in the form

$$u = u_0 \sin(-\omega t) \tag{467}$$

the length of the, for instance, semi-major tidal axis Δx can be obtained by integrating (467) over one quarter of a tidal period T, from $t = 0$ to $t = T/4$. With $u = dx/dt$,

$$\int_{t=0}^{T/4} \frac{dx}{dt}\, dt = \int_{t=0}^{T/4} u_0 \sin(-\omega t)\, dt \tag{468}$$

Thus,

$$x_{t=T/4} - x_{t=0} \equiv \Delta x = \frac{u_o}{\omega} [\cos(-\omega t)]_0^{T/4} = \frac{u_0}{\omega}\, [0 - 1] \tag{469}$$

where $\omega = 2\pi/T$ has been used in the last equality. Therefore,

$$|\Delta x| = u_0 \frac{T}{2\pi} \tag{470}$$

For the $\mathbf{M_2}$ tide, $T_{M_2} = 12.42$ hr. In this case,

$$\Delta x_{M_2} \approx (7 \cdot 10^3 \text{ s})\, u_0 \tag{471}$$

For moderately to very strong $\mathbf{M_2}$ tidal currents, u_0 ranges from 0.1 m/s to 1 m/s, respectively, and the corresponding lengths of the semi-major axes are

$$\Delta x_{M_2} \approx 0.7 \text{ km} \quad \text{and} \quad \Delta x_{M_2} \approx 7 \text{ km} \tag{472}$$

The particle trajectories describing the elliptic path are orders of magnitude smaller than the typical wavelength of several 100 km to a few 1000 km for the tidal waves, see Sec. 20.1.1. It is, however, the co-ordinated movement of the small-scale tidal ellipses that generates the large-scale progression of the tidal wave. Put differently, the progression of the fast, large-scale tidal wave seen as variations in the surface elevation ζ is *not* associated with advection of mass, but it is a consequence of the coordinated horizontal, smaller-scale, elliptic trajectories of water packages throughout the water column. The latter movements *do* represent advection of mass. Passive particles released in the ocean, or e.g. buoyant neutral fish eggs or larvae, will therefore, in the absence of other processes and forces, span out an elliptic trajectory during a tidal period, i.e., approximately every 12 hr or 24 hr depending on the tidal regime.

21.3.3 Case $f = 0$

In the case of no rotation, $f = 0$, $\omega = \pm\sqrt{gD}\, k$ and the phase speed $c = \omega/k = \pm\sqrt{gD}$. This is the solution of gravity waves.

21.3.4 Case $k^2 \gg f^2/(gD)$ (short waves)

The dispersion relationship (447) can be put in the form

$$\omega = \pm\sqrt{gD \left(\frac{f^2}{gD} + k^2\right)} \tag{473}$$

For $k^2 \gg f^2/(gD)$, implying waves with short wavelengths, $\omega \approx \pm\sqrt{gD}\, k$ and the phase speed $c = \omega/k \approx \pm\sqrt{gD}$. Also this solution regime represents the gravity waves. The reason for this is that waves with wavelengths much shorter than the deformation radius L_ρ are only weakly influenced by the rotation of the Earth.

21.3.5 Case $k^2 \ll f^2/(gD)$ (long waves)

In the case of (very) long wavelengths, e.g. for $k^2 \ll f^2/(gD)$, $\omega \to \pm f$ (see 473). This case represents large-scale, horizontal oscillations with the frequency of the Coriolis parameter f. These are the inertial oscillations or waves.

For $\omega = f$ and since k is small (i.e., $k \to 0$), expression (437) gives

$$u = i\,v \tag{474}$$

Consequently,

$$u = \mathrm{Re}\{iv_0 \exp(-ift)\} = \mathrm{Re}\{iv_0[\cos(-ft) + i\sin(-ft)]\} = -v_0\sin(-ft) \tag{475}$$

and

$$v = \mathrm{Re}\{v_0 \exp(-ift)\} = \mathrm{Re}\{v_0[\cos(-ft) + i\sin(-ft)]\} = v_0\cos(-ft) \tag{476}$$

On the northern hemisphere with $f > 0$, expression (475) and (476) describe a circular, horizontal motion with radius v_0 and rotation directed *with* the clock (or anti-cyclonic rotation). These are the *inertial oscillations* in the ocean.

21.4 Interaction between tidal ellipses and internal oscillations

From Sections 21.3.1 and 21.6, it follows that tidal ellipses and inertial motion describe similar, rotational movement. Thus, if the period of any of the (major) tidal constituents are similar to the period of the inertial oscillations, strong interactions (resonance) are expected. This is indeed the case at specific geographic latitudes ϕ, known as *critical latitudes*. **/p. 126/**

If we assume that the temporal evolution of the horizontal velocity component u of a tidal ellipsis can be written as

$$u \propto \exp(-i\,\omega t) \tag{477}$$

the local acceleration term in the momentum equation, $\partial u/\partial t$, scales as

$$\omega\,u \tag{478}$$

Similarly, the momentum equation's Coriolis-term $f\,\hat{\mathbf{z}} \times \mathbf{u}$ scales as

$$f\,u \tag{479}$$

Here $f = 2\Omega\sin\phi$ is the Coriolis parameter, Ω is the Earth's rotation vector ($\Omega = 2\pi/(24\,\mathrm{msh})$), and $\hat{\mathbf{z}}$ is the outward directed unit vector.

For the $\mathbf{M_2}$ tide, the speed follows from Tables 2 and 6:

$$\omega = 2\,(\omega_s - \omega_2) = 2\,\omega_1 \tag{480}$$

where $\omega_1 = 0.2529$ rad/msh. Consequently, $\omega = f$ occurs for **/p. 126/**

$$\sin\phi = \frac{\omega}{2\,\Omega}\,, \qquad \text{or for} \qquad \phi = \pm 75^\circ \tag{481}$$

Thus, strong interaction between the $\mathbf{M_2}$ tide and inertial oscillations are expected (and indeed observed) at 75°N, like in the Barents Sea, and at 75°S in the Southern Ocean's Weddel and Ross Seas.

Also two times the $\mathbf{M_2}$ tidal period may interact with the inertial oscillations. This gives rise to the critical latitudes

$$\phi = \pm 29^\circ \tag{482}$$

For the $\mathbf{K_1}$ tide with frequency $\omega = \omega_1 + \omega_2 = 0.2622$ rad/msh (see Tables 2 and 6), the critical latitudes are **/p. 126/**

$$\phi = \pm 30^\circ \tag{483}$$

The critical northern latitudes for $\mathbf{M_2}$ and $\mathbf{K_1}$ are displayed in figure 33, together with the period (in hr) of the Coriolis parameter f. Similar relationship exists for the southern hemisphere.

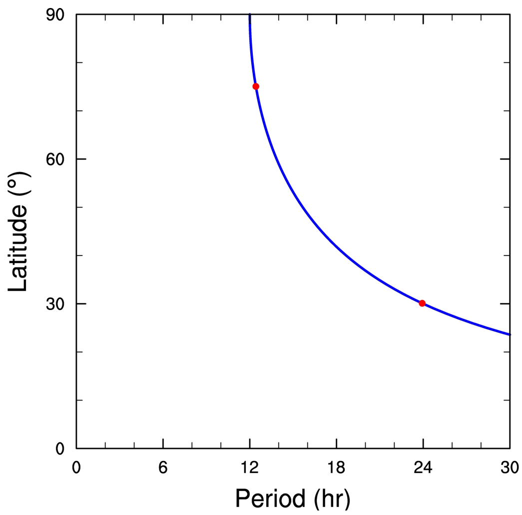

Figure 33: The latitudinal variation of the period (hr) of the Coriolis parameter f in blue, together with the period of the $\mathbf{M_2}$ and $\mathbf{K_1}$ tidal constituents (red dots) with critical latitude of 75° and 30°, respectively.

Part V

Appendix

A Some key parameters of the Earth, Moon, Sun system

A.1 Mass

The mass of the Earth m_e, Moon m_l and Sun m_s are

$$m_e = 5.9722 \cdot 10^{24} \text{ kg} \quad (484)$$
$$m_l = 7.35 \cdot 10^{22} \text{ kg} \quad (485)$$
$$m_s = 1.9884 \cdot 10^{30} \text{ kg} \quad (486)$$

The following mass ratios are then obtained

$$\frac{m_e}{m_l} \approx 80\,, \qquad \frac{m_l}{m_e} \approx 0.012 \quad (487)$$
$$\frac{m_e}{m_s} \approx 3 \times 10^{-6}\,, \qquad \frac{m_s}{m_e} \approx 3 \times 10^{5} \quad (488)$$
$$(489)$$

A.2 Length and distance

Earth's mean radius is

$$|\mathbf{r}| = 6.378 \cdot 10^6 \text{ m} \quad (490)$$

Often a, in stead of $|\mathbf{r}|$, is used to denote the Earth's mean radius.

The mean distance between the Earth and the Moon, and the Earth and the Sun, are

$$R_{l,\,\text{mean}} = 3.844 \cdot 10^8 \text{ m} \quad (491)$$
$$R_{s,\,\text{mean}} = 1.496 \cdot 10^{11} \text{ m} \quad (492)$$

The following distance ratios are obtained

$$\frac{a}{R_{l,\,\text{mean}}} \approx 0.017\,, \qquad \frac{R_{l,\,\text{mean}}}{r} \approx 60 \quad (493)$$
$$\frac{a}{R_{s,\,\text{mean}}} \approx 4 \times 10^{-5}\,, \qquad \frac{R_{s,\,\text{mean}}}{r} \approx 2 \times 10^{4} \quad (494)$$
$$(495)$$

B Spherical coordinates

B.1 Two commonly used spherical coordinate systems

Spherical coordinates are, by construction, convenient for any type of spherical problems. Fig. 34 illustrates two commonly used variants of spherical coordinates.

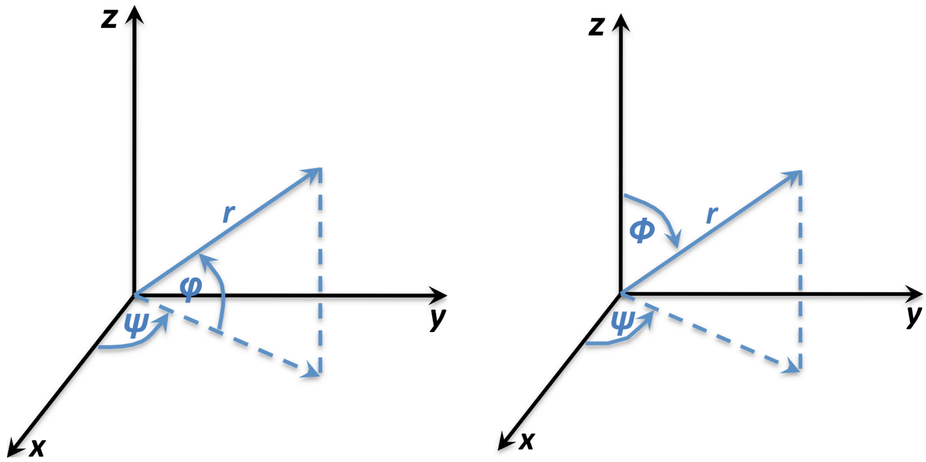

Figure 34: Illustration of two spherical coordinate systems.

The system to the left is described by Ψ (angle in longitudinal direction, $0 \geq \Psi \geq 2\pi$), φ (angle in the latitudinal direction, $-\pi/2 \geq \varphi \geq \pi/2$) and r (length in the radial direction, $r \geq 0$). This coordinate system is commonly used in oceanography and meteorology because of the correspondence with the commonly used Earth's latitude and longitude positions.

The system to the right is described by Ψ (azimuth angle, $0 \geq \Psi \geq 2\pi$), ϕ (zenith angle, $0 \geq \phi \geq \pi$) and r (length in the radial direction, $r \geq 0$). This coordinate system is commonly used in problems involving gravitational or electric potential as will become clear shortly.

Note that the set of coordinates in the order $(\Psi\,,\ \varphi\,,\ r)$ for the system to the left in Fig. 34, and $(r\,,\ \phi\,,\ \Psi)$ for the system to the right, are both right-hand coordinate systems. Also note that the actual naming of the various angels may vary depending on the problem; the azimuth angle Ψ is, as an example, commonly labelled λ in meteorology and oceanography.

B.2 Volume and surface elements in spherical coordinates

B.2.1 Spherical volume elements

The volume element formed by perturbing each of the three coordinates, denoted by δ in the following, can be obtained as follows:

For the system to the left in Fig. 34, small changes in r form a line segment of length δr in the direction of $\mathbf{r}$; small changes in φ form an upward directed arc of length $r\,\delta\varphi$; and small changes in Ψ form an arc in the xy-plane of length $r\cos\varphi\,\delta\Psi$. The resulting volume element is the product of the three length contributions

$$\delta V_{\Psi\varphi r} = r^2 \cos\varphi\,\delta r\,\delta\varphi\,\delta\Psi \tag{496}$$

Correspondingly, for the system to the right in the figure, small changes in r form a line segment of length δr in the direction of $\mathbf{r}$, small changes in ϕ form an downward directed arc of length $r\,\delta\phi$, and small changes in Ψ form an arc in the xy-plane of length $r\sin\phi\,\delta\Psi$. The resulting volume element is

$$\delta V_{r\phi\Psi} = r^2 \sin\phi\,\delta r\,\delta\phi\,\delta\Psi \tag{497}$$

B.2.2 Spherical surface elements

If r is kept constant at $r = a$, the surface area S of the sphere spanned out by the two pairs of angles $(\Psi,\ \varphi)$ and $(\phi,\ \Psi)$ is given the volume elements above when changes in r are ignored

$$\delta S_{\Psi\varphi} = a^2 \cos\varphi\,\delta\varphi\,\delta\Psi \tag{498}$$

$$\delta S_{\phi\Psi} = a^2 \sin\phi\,\delta\phi\,\delta\Psi \tag{499}$$

Integration over the full sphere, i.e.,

$$\int_{\Psi=0}^{2\pi}\int_{\varphi=-\pi/2}^{\pi/2} \delta S_{\Psi\varphi}\,d\varphi\,d\Psi \tag{500}$$

$$\int_{\Psi=0}^{2\pi}\int_{\phi=0}^{\pi} \delta S_{\phi\Psi}\,d\phi\,d\Psi \tag{501}$$

result in the surface area of a sphere with radius a, namely $4\pi a^2$, as expected.

B.2.3 Relationship with the Earth-Moon system

Note that in the system to the right in Fig. 34, the zenith angle ϕ is always non-negative. For $r = \text{const}$ and $0 \le \Psi < 2\pi$, the resulting geometry forms a cone with angle $0 \le \phi \le \pi$ centred around the positive z-axis. A cone is also spanned out for $r = \text{const}$, $\varphi = \text{const}$ and $0 \ge \Psi \ge 2\pi$ in the system to the left in the figure. In the latter case, however, the latitudinal angle $-\pi/2 \le \varphi \le \pi/2$ takes both signs.

It is the non-negative zenith-angle ϕ in the coordinate system to the right in Fig. 34, similar to the zenith angle ϕ in tidal theory when the centre line in Fig. 4 is aligned with the z-axis in Fig. 34, that makes the (ϕ,Ψ)-system particularly suited for problems involving gravitational potential (or problems involving electric potentials), see e.g. Sec. 5.10.1.

B.2.4 Gradient operator in spherical coordinates

The gradient operator expressed in terms of the two spherical coordinate systems in Fig. 34 can be expressed in terms of the two system's scale factors.

For the system to the left in Fig. 34, the scale factors in the Ψ,φ,r-directions are $r\cos\varphi$, r and 1, respectively. The resulting gradient operator is therefore

$$\nabla = \left(\frac{1}{r\cos\varphi}\frac{\partial}{\partial\Psi}, \frac{1}{r}\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial r}\right) = \frac{1}{r\cos\varphi}\frac{\partial}{\partial\Psi}\,\mathbf{e}_\Psi + \frac{1}{r}\frac{\partial}{\partial\varphi}\,\mathbf{e}_\varphi + \frac{\partial}{\partial r}\,\mathbf{e}_r \tag{502}$$

In the above expression, $\mathbf{e}_\Psi$, $\mathbf{e}_\varphi$ and $\mathbf{e}_r$ are the unit vectors in the Ψ,φ,r-directions, respectively.

Similarly, for the right system in Fig. 34, the scale factors in the r, ϕ, Ψ-directions are 1, r and $r \sin\phi$, respectively. The resulting gradient operator becomes

$$\nabla = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \phi}, \frac{1}{r \sin\phi}\frac{\partial}{\partial \Psi} \right) = \frac{\partial}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial \phi}\,\mathbf{e}_\phi + \frac{1}{r \sin\phi}\frac{\partial}{\partial \Psi}\,\mathbf{e}_\Psi \tag{503}$$

C The three lunar tidal components

To show the identity between the right hand sides of (69) and (70), one can first note that $\overline{\zeta}_1$ is common to both expressions and can therefor be ignored in the following.

Expansion and reordering of $\overline{\zeta}_0$ gives

$$\begin{aligned}\overline{\zeta}_0 &= \frac{3}{2}\left(\sin^2\phi - \frac{1}{3}\right)\left(\sin^2 d - \frac{1}{3}\right)\\ &= \frac{3}{2}\sin^2\phi\,\sin^2 d - \frac{1}{2}\sin^2\phi - \frac{1}{2}\sin^2 d + \frac{1}{6}\\ &= \sin^2\phi\,\sin^2 d + \frac{1}{2}\sin^2\phi\,\sin^2 d - \frac{1}{2}\sin^2\phi - \frac{1}{2}\sin^2 d + \frac{1}{6}\\ &= \sin^2\phi\,\sin^2 d + \frac{1}{2}\sin^2\phi\,(\sin^2 d - 1) - \frac{1}{2}\sin^2 d + \frac{1}{6}\\ &= \sin^2\phi\,\sin^2 d - \frac{1}{2}\sin^2\phi\,\cos^2 d - \frac{1}{2}\sin^2 d + \frac{1}{6} \qquad (504)\end{aligned}$$

In the second last line, the identity

$$\sin^2 a + \cos^2 a = 1 \qquad (505)$$

has been used.

Furthermore

$$\begin{aligned}\overline{\zeta}_2 &= \frac{1}{2}\cos^2\phi\,\cos^2 d\,\cos(2C_P) \qquad (506)\\ &= \cos^2\phi\,\cos^2 d\,\cos^2 C_P - \frac{1}{2}\cos^2\phi\,\cos^2 d \qquad (507)\end{aligned}$$

Here the identity

$$\cos 2a = \cos^2 a - \sin^2 a = 2\,\cos^2 a - 1 \qquad (508)$$

has been applied to the last factor in (506).

With $\overline{\zeta}_0$ from (504) and $\overline{\zeta}_2$ from (507), one obtains

$$\begin{aligned}\overline{\zeta}_0 + \overline{\zeta}_2 &= \sin^2\phi\,\sin^2 d - \frac{1}{2}\sin^2\phi\,\cos^2 d - \frac{1}{2}\sin^2 d + \frac{1}{6}\\ &\quad + \cos^2\phi\,\cos^2 d\,\cos^2 C_P - \frac{1}{2}\cos^2\phi\,\cos^2 d \qquad (509)\end{aligned}$$

The second and last term on the right hand side can be combined into

$$-\frac{1}{2}\cos^2 d \qquad (510)$$

so

$$\overline{\zeta}_0 + \overline{\zeta}_2 = \sin^2\phi\,\sin^2 d - \frac{1}{2}\cos^2 d - \frac{1}{2}\sin^2 d + \frac{1}{6} + \cos^2\phi\,\cos^2 d\,\cos^2 C_P \qquad (511)$$

Comparison between the right hand sides of (69) and (511), and remembering that $\overline{\zeta}_1$ has already been taken care of, shows that the two last terms on the right hand side of (511) have their counterparts on the right hand side of (69). For the remaining terms on the right hand side of (511), one obtains

$$\sin^2\phi\,\sin^2 d - \frac{1}{2}\cos^2 d - \frac{1}{2}\sin^2 d + \frac{1}{6} = \sin^2\phi\,\sin^2 d - \frac{1}{2} + \frac{1}{6} = \sin^2\phi\,\sin^2 d - \frac{1}{3} \qquad (512)$$

which is identical to the remaining terms on the right hand side of (69).

D Elliptic geometry[19]

An ellipse is the locus of points M in the plane such as the sum of the distances MF and MF' to two fixed points F and F', called the *foci*, is constant. A basic configuration is shown in Figure 35.

From the definition, an ellipsis can be formed by fixing a non-stretching string of length l at two sticks (the foci points), a distance $d < l$ apart. A third moveable stick will then form the ellipse if the string runs on the outer side of the latter, while the stick is continuously moved with the string fully stretched.

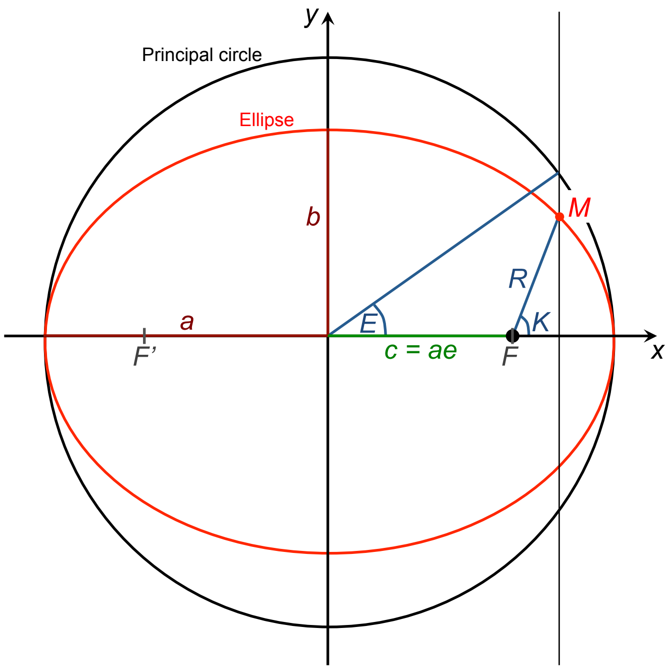

Figure 35: Illustration of an ellipse. a and b are the *semi-major* and *semi-minor* axes, respectively; F and F' are the *foci points* (of which a point with mass is located in foci point F whereas F' is an empty focus (i.e., no mass)); M is a point on the ellipse; R and K are the distance FM and the angle angle between the x-axis and the line segment R, respectively; c is the distance between the centre of the ellipse and the foci points; and e is the *eccentricity* of the ellipse satisfying $c = a\,e$, with $0 \leq e < 1$. The angles K and E are called the *true anomaly* and the *eccentric anomaly*, respectively. Note that the eccentric anomaly defines the angle between the horizontal axis and a point on the principal circle given by a vertical line running through M.

As seen from Fig. 35, the eccentricity e is defined by the relationship

$$c = a\,e\,, \qquad 0 \leq e < 1 \tag{513}$$

[19]Mainly based on the excellent treatise by M. Capderou (2014): *Handbook of Satellite Orbits. From Kepler to GPS*, ISBN 978-3-319-03415-7, DOI 10.1007/978-3-319-03416-4, Springer International Publishing Switzerland.

Let

$$R = F\,M \qquad \text{and} \qquad R' = F'\,M \tag{514}$$

In the case $K = 0$, it follows that

$$R = a - c \qquad \text{and} \qquad R' = a + c \tag{515}$$

Therefore,

$$R + R' = 2\,a \tag{516}$$

or that the sum of $R + R'$ is constant, consistent with the definition. That the constant equals $2\,a$ follows directly from Fig. 35 in the case $K = 0$, but is otherwise not intuitively given.

Another property is obtained when $K = \pi/2$. In this case

$$R = R' = a \tag{517}$$

(from 516), and Pythagoras gives

$$b^2 = a^2 - c^2 = a^2\,(1 - e^2) \tag{518}$$

In the last equality, the definition (522) is being used. Thus,

$$\frac{b}{a} = \sqrt{1 - e^2} \tag{519}$$

D.1 Cartesian coordinates

In Cartesian coordinates, M is located at the point (x, y) and F is located at $(c, 0)$. Pythagoras gives

$$R^2 = (x - c)^2 + y^2 = (x - ae)^2 + y^2 \tag{520}$$

Likewise,

$$R'^2 = (x + c)^2 + y^2 = (x + ae)^2 + y^2 \tag{521}$$

Therefore

$$R'^2 - R^2 = 4\,cx = 4\,ae\,x \tag{522}$$

Furthermore

$$R'^2 - R^2 = (R' - R)(R' + R) \tag{523}$$

and by means of (516), we obtain

$$R' - R = 2\,ex \tag{524}$$

Finally, by combining (516) and (524), we get the lengths of R and R' expressed in Cartesian coordinates

$$R = a - ex \tag{525}$$
$$R' = a + ex \tag{526}$$

The standard Cartesian formula for an ellipse is obtained by combining (520) and (525):

$$R^2 = (x + ae)^2 + y^2 = (a - ex)^2 \tag{527}$$

giving

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1 \tag{528}$$

or, by means of (518),

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{529}$$

D.2 Semilatus rectum

The distance p from the x-axis to any point y on the ellipsis is called the ellipse's semilatus rectum, see Fig. 36.

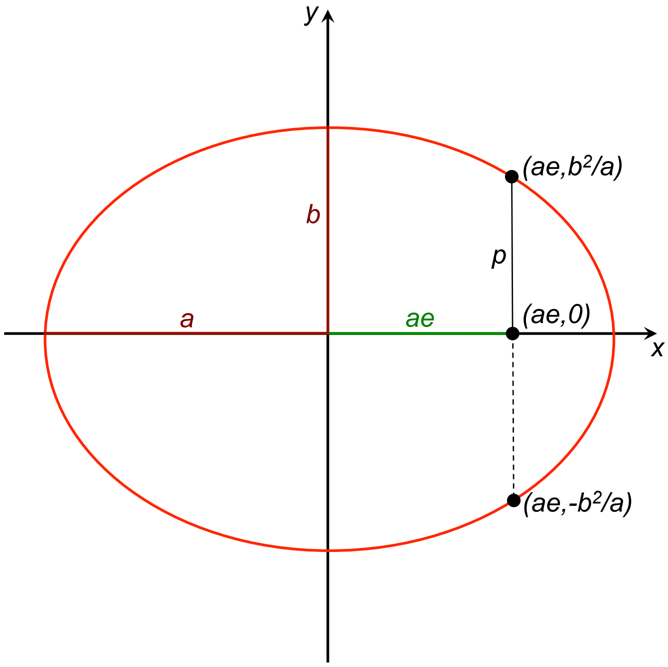

Figure 36: Illustration of the semilatus rectum p of an ellipse, the latter with semi-major and semi-minor axes a and b, respectively, and eccentricity e.

The semilatus rectum can be derived from expression (529), with $x = ae$:

$$\frac{y^2}{b^2} = 1 - e^2 \tag{530}$$

But from expression (519)

$$1 - e^2 = \frac{b^2}{a^2} \tag{531}$$

therefore

$$y = \pm\frac{b^2}{a} \tag{532}$$

and the semilatus rectum is therefore $p = b^2/a$ ($a > b$) .

Furthermore, the ratio b/a given by (519) implies that p can be expressed as

$$p = a(1 - e^2) \tag{533}$$

D.3 The y-value of an ellipse relative to the reference circle

On a reference circle, the positive y value at $x = ae$ is given by Pythagoras

$$y_{\text{circle}} = a\sqrt{1-e^2} \tag{534}$$

On an ellipse, the corresponding y-value is given by expression (530)

$$y_{\text{ellipse}} = b\sqrt{1-e^2} \tag{535}$$

Therefore, for any x-value, the ratio of the y-value of the ellipse and the reference circle is constant and is given by

$$\frac{y_{\text{ellipse}}}{y_{\text{circle}}} = \frac{b}{a} = \sqrt{1-e^2} \tag{536}$$

where the last equality comes from expression (519).

D.4 Polar coordinates

We first define a Cartesian coordinate system centred on the foci point F. The x-coordinate of F, labelled X, is

$$X = x - c = x - ea \tag{537}$$

The length R, from (525), can now be expressed as

$$R = a - ex = a - e(X + ea) = a(1-e^2) - eX \tag{538}$$

From Fig. 35,

$$X = R\cos K \tag{539}$$

Inserting into (538) and rearranging yields the equation for an ellipse in polar coordinates

$$R(K) = \frac{a(1-e^2)}{1 + e\cos K} \tag{540}$$

The inverse of R can be expressed with one term proportional to $\cos K$ plus a constant term:

$$\frac{1}{R(K)} = \frac{e}{a(1-e^2)}\cos K + \frac{1}{a(1-e^2)} \tag{541}$$

Thus, expressions (540) and (541) are the polar counterparts to the more commonly known cartesian expression given by (529).

E Miscellaneous notes

E.1 Including variations in the lunar distance

The lunar distance R_l in the Equilibrium Tide, see expressions (74)–(77), is not constant, but varies according to the Moon's elliptic path around the Earth. The latter has the form of the varying Sun–Earth distance, see expression (282), but expressed with the lunar subscript l, the *mean ecliptic longitude* s and the *longitude of the lunar perigee* p: **/3.18/**

$$\frac{\overline{R}_l}{R_l} = 1 + e\cos(s-p) \tag{542}$$

Including varying lunar distance in the semi-diurnal component C_2, see expression (77), gives **/p. 63/**

$$\begin{aligned} C_2(t) &= \left[\left(\frac{a}{\overline{R}_l}\right)^3 \frac{3}{4}\cos^2 d_l\right] [1 + e\cos(s-p)]^3 \cos 2\, C_l \\ &= \Psi\, [1 + e\cos(s-p)]^3 \cos 2\, C_l \end{aligned} \tag{543}$$

where

$$\Psi = \left[\left(\frac{a}{\overline{R}_l}\right)^3 \frac{3}{4}\cos^2 d_l\right] \tag{544}$$

has been included for convenience.

Due to the smallness of e, the binomial formula gives

$$[1 + e\cos(s-p)]^3 \approx 1 + 3\, e\cos(s-p) \tag{545}$$

Furthermore,

$$\begin{aligned} \cos 2\, C_l &= \cos 2\, [\omega_0 t + h - s - \pi - 2\, e\sin(s-p)] \\ &= \cos 2\, [\delta - 2\, e\sin(s-p)] \end{aligned} \tag{546}$$

where

$$\delta \equiv \omega_0 t + h - s - \pi \tag{547}$$

Expression (546) can be simplified, again by using the smallness of e:

$$\begin{aligned} \cos 2\, C_l &= \cos 2\, [\delta - 2\, e\sin(s-p)] \\ &= \cos\, [2\,\delta - 4\, e\sin(s-p)] \\ &\overset{(620)}{=} \cos 2\,\delta\cos([4\, e\sin(s-p)] + \sin 2\,\delta\sin([4\, e\sin(s-p)] \\ &\approx \cos 2\,\delta + 4\, e\sin 2\,\delta\sin(s-p) \end{aligned} \tag{548}$$

where (639) and (640) are used in the last equality.

With expression (545) and (548) inserted into (543), we obtain

$$C_2(t) = \Psi\, [\cos 2\,\delta + 4\, e\sin 2\,\delta\, \sin(s-p) + 3\, e\cos(s-p)\, \cos 2\,\delta\,] \tag{549}$$

From the identities (621) and (616), we get

$$4\, e\sin 2\,\delta\, \sin(s-p) = 2\, e\cos(2\delta - s + p) - 2\, e\cos(2\delta + s - p) \tag{550}$$

Similarly, the identities (622) and (616) give

$$3\,e\cos(s-p)\,\cos 2\,\delta = \frac{3}{2}\,e\cos(2\delta - s + p) + \frac{3}{2}\,e\cos(2\delta + s - p) \tag{551}$$

Consequently, **/4.3/**

$$\boxed{C_2(t) = \Psi\left[\cos 2\delta + \frac{7}{2}\,e\cos(2\delta - s + p) - \frac{3}{2}\,e\cos(2\delta + s - p)\right]} \tag{552}$$

In the above expression, the first term is $\mathbf{M_2}$ with speed $2\,\sigma_1$. The second and third terms have speeds $2\,\sigma_1 - \sigma_2 + \sigma_4$ and $2\,\sigma_1 + \sigma_2 - \sigma_4$, respectively. Since the two latter have speeds close to – and symmetric about – the speed of $\mathbf{M_2}$, these harmonics are named $\mathbf{N_2}$ (the *larger elliptical lunar*) and $\mathbf{L_2}$ (*the smaller elliptical lunar*), respectively. **/p. 63/**

E.2 High and low water times and heights

E.2.1 High water times and heights

We consider the height $T(t)$ of the major $\mathbf{M_2}$ tide and another minor tidal constituent: **/p. 368/**

$$T(t) = H_{M_2}\cos\omega t + A\cos nt + B\sin nt \tag{553}$$

The latter minor constituent can be cast into the form of a single trignometric function by introducing the amplitude R and the angle θ:

$$A = R\,\cos\theta \tag{554}$$

$$B = R\,\sin\theta \tag{555}$$

The above equations relate A, B and R, θ:

$$R^2 = A^2 + B^2\,, \qquad \text{and} \qquad \tan\theta = \frac{B}{A} \tag{556}$$

Thus, by means of expression (620), equation (553) can also be written as

$$T(t) = H_{M_2}\cos\omega t + R\,\cos(nt - \theta) \tag{557}$$

Maximum value of (553) occurs for a vanishing temporal derivative:

$$\frac{\partial T(t)}{\partial t} = -\omega H_{M_2}\sin\omega t - nA\sin nt + nB\cos nt = 0 \tag{558}$$

If $t = 0$ denotes the time for maximum $\mathbf{M_2}$, we can use for small t:

$$\cos\omega t \approx \cos nt \approx 1\,, \qquad \sin\omega t \approx \omega t\,, \qquad \text{and} \qquad \sin nt \approx nt \tag{559}$$

The maximum amplitude $H_{\max}$ of the combined tide occurs when the time derivative in (558) vanishes:

$$-\omega^2 t_{\max}\,H_{M_2} - n^2 t_{\max}\,A + n\,B = 0 \tag{560}$$

As long as $\mathbf{M_2}$ is the leading tidal constituent, $R/H_{M_2} \ll 1$, giving

$$t_{\max} = \frac{B\,n}{H_{M_2}\,\omega^2} \tag{561}$$

With $t_{\max}$ inserted into (553), and by using

$$\cos\omega t_{\max} \approx 1 - \frac{1}{2}(\omega t_{\max})^2\,, \qquad \cos n t_{\max} \approx 1\,, \qquad \sin n t_{\max} \approx n t_{\max} \tag{562}$$

we obtain

$$H_{\max} = H_{M_2} + A + \frac{1}{2}\left(\frac{n}{\omega}\right)^2 \frac{B^2}{H_{M_2}} \tag{563}$$

E.2.2 Low water times and heights

Likewise, if we let $t = 0$ occur at minimum $\mathbf{M_2}$, expression (553) reads

$$T(t) = -H_{M_2}\cos\omega t + A\cos n t + B\sin n t \tag{564}$$

With the same simplifications as above, we obtain

$$t_{\min} = -\frac{B\,n}{H_{M_2}\,\omega^2} \tag{565}$$

With $t_{\min}$ inserted into (564), and by using

$$\cos\omega t_{\min} \approx 1 - \frac{1}{2}(\omega t_{\min})^2\,, \qquad \cos n t_{\min} \approx 1\,, \qquad \sin n t_{\min} \approx n t_{\min} \tag{566}$$

we get

$$H_{\min} = -H_{M_2} + A - \frac{1}{2}\left(\frac{n}{\omega}\right)^2 \frac{B^2}{H_{M_2}} \tag{567}$$

E.2.3 Temporal average

The temporal averaged value or any quantity Ψ, denoted $\overline{\Psi}$, averaged over many tidal periods $m\,T_{\mathrm{t}}$, are given by

$$\overline{\Psi} = \frac{1}{m\,T_{\mathrm{t}}}\int_0^{m\,T_{\mathrm{t}}} \Psi\,dt \tag{568}$$

The temporal average of $H_{\max}$ from (563) with A, B given by (555), gives the following for the first term

$$\overline{H_{\max}} = \frac{1}{m\,T_{\mathrm{t}}}\int_{t=0}^{m\,T_{\mathrm{t}}} H_{\max}\,dt = \frac{H_{\max}}{m\,T_{\mathrm{t}}}\int_{t=0}^{m\,T_{\mathrm{t}}} dt = H_{\max} \tag{569}$$

The second terms gives no contribution due to the symmetric cosine-contribution around zero:

$$\overline{A} = \frac{1}{m\,T_{\mathrm{t}}}\int_{t=0}^{m\,T_{\mathrm{t}}} A\,dt = \frac{1}{m\,2\pi}\int_{\theta=0}^{m\,2\pi} R\,\cos\theta\,d\theta = 0 \tag{570}$$

For the third term, the $\sin^2$-dependency gives a contribution due to it's non-negative values:

$$\overline{B^2} = \frac{1}{m\,2\pi}\int_{\theta=0}^{m\,2\pi} R^2\,\sin^2\theta\,d\theta \overset{(637)}{=} \frac{R^2}{m\,2\pi}\left[\frac{\theta}{2} - \frac{\sin\,2\theta}{4}\right]_{\theta=0}^{m\,2\pi} = \frac{R^2}{2} \tag{571}$$

Consequently, from (563),

$$\overline{H_{\max}} = H_{\max} + \frac{1}{4}\left(\frac{n}{\omega}\right)^2 \frac{R^2}{H_{M_2}} \tag{572}$$

Likewise, the temporal average of $H_{\min}$, from (567), becomes

$$\overline{H_{\min}} = -H_{\min} - \frac{1}{4}\left(\frac{n}{\omega}\right)^2 \frac{R^2}{H_{M_2}} \tag{573}$$

Thus, the Mean High Water (MHW) and Mean Low Water (MLW) are given by **/C.1/**

$$\text{MHW} = \overline{H_{\max}} \tag{574}$$
$$\text{MLW} = \overline{H_{\min}} \tag{575}$$

The Mean Tide Level (MTL) is the mean of MHW and MLW, so

$$\text{MTL} = 0 \tag{576}$$

Finally, the Mean Tidal Range (MTR) is the difference between MHW and MLW,

$$\text{MTR} = \text{MHW} - \text{MLW} = 2\,H_{\max} + \frac{1}{2}\left(\frac{n}{\omega}\right)^2 \frac{R^2}{H_{M_2}} \tag{577}$$

E.3 Polar tide

If ω and $\mathbf{r}$ are the Earth's rotation and radius vectors, respectively, the centrifugal acceleration due to Earth's rotation can be written as

$$-\omega \times (\omega \times \mathbf{r}) = \omega^2\, R\,(0, -\sin\varphi, \cos\varphi) \tag{578}$$

Here $R = r\cos\varphi$ is the distance from the rotation axis to a point A on Earth's surface and φ is the latitude of A.

Alternatively, R can be expressed in terms of the colatitide θ,

$$R = r\,\sin\theta \tag{579}$$

The centrifugal acceleration can be written as the gradient to a scalar, the centrifugal potential V:

$$-\omega \times (\omega \times \mathbf{r}) = \nabla\left(\frac{\omega^2 R^2}{2}\right) = \nabla V \tag{580}$$

Thus, **/p. 260/**

$$V = \frac{1}{2}\,\omega^2 r^2\,\sin^2\theta \tag{581}$$

A small change in the colatitude angle $\delta\theta$ will lead to a change δV in the centrifugal potential accroding to the expression

$$\delta V = \frac{1}{2}\,\omega^2 r^2\, 2\,\sin\theta\cos\theta\,\delta\theta \overset{(619)}{=} \frac{1}{2}\,\omega^2 r^2\,\sin 2\theta\,\delta\theta \tag{582}$$

Thus, perturbations in θ, for instance caused by polar motion, give rise to largest changes in the centrifugal acceleration at $\pm 45°$, and vanishing changes at equator and at the poles. **/Fig. 10.5/**

F Some wave characteristics

A wave in the ocean (or in the atmosphere or in a fluid) can be described as

> *a physical processes that transport information in time and space, such as energy, without or with little advection of mass associated with the transport, and with speed and direction that is generally different from the advection of mass (or the ocean general circulation).*

F.1 Properties of waves in one spatial dimension

Any perturbation (small change) can be expressed as the sum of trigonometric (sine or cosine) waves, each having a specific amplitude, wavelength, period, and phase.

F.1.1 Spatial variation

A stationary wave in the x-direction, for example expressed as the sea level elevation ζ (m), can be expressed as

$$\zeta(x) = a \cos\left[2\pi \frac{x}{\lambda_x}\right] \tag{583}$$

or a sum of waves of similar form.

The wave elevation ζ is characterised by

Amplitude a : Implying that ζ varies between $\pm a$. Unit is metre.

Wavelength λ_x : Implying that the wave repeats itself when $x = \pm n\,\lambda_x$, where n is an integer. Unit is metre.

F.1.2 Spatial-temporal variations

A wave will normally propagate in time. The wave's

Speed, alternatively it's phase speed is denoted c_x, see also *phase speed* below. The wave can propagate in the positive and the negative x-direction. Thus, the phase speed is $\pm c_x$, with the convention that $c_x > 0$. Alternatively, c_x may take both positive and negative values without explicit notation of the sign. Unit is m s^{-1}.

Since speed = distance/time, the wave propagates a distance $x' = \pm c_x t$ in time t. The wave form (583) can then be written as

$$\zeta(x,t) = a \cos\left[\frac{2\pi}{\lambda_x}(x \pm c_x t)\right] \tag{584}$$

The above wave is characterised by

The phase given by the argument $2\pi(x \pm c_x t)/\lambda_x$ (in radians). Points with constant phase are points where the wave form has the same value, like the wave crest or the wave through.

Positive sign in front $c_x t$ in (584) describes a wave propagating in the *negative* x-direction. This is seen from $2\pi(x + c_x t)/\lambda_x = C$, where C is a constant (i.e., we consider a constant phase). Therefore, $x = C\lambda_x/(2\pi) - c_x t$. Since $\lambda_x, c_x > 0$, this implies decreasing x with increasing t. Similarly, negative sign in front of $c_x t$ gives rise to wave propagation in the positive x-direction.

F.1.3 Wave number

Instead of using wavelength λ_x, it is common to express the wave with the

Wavenumber k_x (unit m^{-1}). The relationship between the wave number and the wavelength is given by

$$k_x = \frac{2\pi}{\lambda_x} \tag{585}$$

The wave number is the number of wavelengths confined by the "length" 2π. For example, a wave with wavelength 100 km has a wave number 6.3×10^{-5} m^{-1}. Therefore, the wave number does not have to be an integer.

Using (585) in (584), the latter can be expressed as

$$\zeta(x,t) = a\cos[k_x(x \pm c_x t)] \tag{586}$$

F.1.4 Other definitions

The period T is the time it takes for a point on the wave to repeat itself. T therefore equals the time it takes for the wave to propagate one wavelength

$$T = \lambda_x / c_x \tag{587}$$

The angular frequency (also called the angular speed) ω is a measure of the temporal variation of the wave

$$\omega = \frac{2\pi}{T} \tag{588}$$

The phase speed c_x is the speed of the wave, for example, the speed of the wave crest or through. Consequently,

$$c_x = \frac{\lambda_x}{T} = \frac{\omega}{} k_x \tag{589}$$

where (585) and (588) in the last equality.

By using ω and k_x, expression (586) be put in the form

$$\zeta(x,t) = a\cos(k_x x \pm \omega t) \tag{590}$$

Expressions (584), (586) and (590) describes the same: For fixed time $t = t_0$, the surface elevation ζ is a wave form in x-direction repeating itself with the wavelength λ_x, that is, the wave form is repeated for $x = \pm n\,\lambda_x$ where n is an integer. Similarly, for a fixed point $x = x_0$, ζ describes a standing wave with time t, i.e., a wave that repeats itself every $t = \pm n\,T$ (n being an integer).

Finally,

The phase of a wave is given by the argument $k_x x \pm \omega t$ and expresses any specific point in the wave cycle. The phase varies from 0 to 2π.

F.2 Two-dimensional waves

The above can be extended to multiple dimensions. In two dimensions, the surface elevation can be expressed as

$$\zeta = a\cos(k_x x + k_y y \pm \omega t) = a\cos(\mathbf{k}\cdot\mathbf{x} \pm \omega t) \tag{591}$$

where

The wave number vector $\mathbf{k} = (k_x, k_y) = k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}}$, with magnitude

$$k^2 = k_x^2 +^2 k_y \tag{592}$$

and direction

$$\hat{\mathbf{k}} = \frac{\mathbf{k}}{k} \tag{593}$$

The phase speed is

$$c = \frac{\omega}{k} \tag{594}$$

and the wavelength is

$$\lambda = \frac{2\pi}{k} \tag{595}$$

with the wave number k given by (592).

F.3 Complex notation

Rather than assuming the wave in the form of a cosine (or sine) wave as described above, it is convenient to express the wave in complex form:

$$a\exp[i(\mathbf{k}\cdot\mathbf{x} \pm \omega t)] \tag{596}$$

The physical meaningful quantity will then be the real part of the complex number, like

$$\mathrm{Re}\,\{a\exp[i(\mathbf{k}\cdot\mathbf{x} \pm \omega t)]\} \tag{597}$$

In the above expression, Re denotes the real part, a is the (complex) wave amplitude, $\mathbf{k} = k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}}$ is the wave number vector in the x- and y-directions, $\mathbf{x} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$ is the position vector and ω is the angular frequency or speed.

Since

$$\exp i\psi = \cos\psi + i\sin\psi \tag{598}$$

expression (597) embed the (standard) wave form

$$a\cos(\mathbf{k}\cdot\mathbf{x} \pm \omega t) \tag{599}$$

It has here been assumed that the amplitude a is real.

The waveform given by (597) is particularly convenient since the derivative of the exponential function equals the function itself, corrected with algebraic coefficients resulting from the use of the chain rule. This implies that derivations are readily substituted by algebraic coefficients:

$$\frac{\partial}{\partial x} \to ik_x\,, \quad \frac{\partial}{\partial y} \to ik_y\,, \quad \text{and} \quad \frac{\partial}{\partial t} \to \pm i\omega \tag{600}$$

This inplies that, for example, the shallow water equations can be expressed as a set of algebraic equations that can be readily analysed. The resulting algebraic expression encompass all possible combinations of wave parameters and physical (environmental) parameters that satisfy the full, continuous set of equations. This, together with a physical interpretation of the wave solution, gives a full description of the waves.

F.4 Dispersion relationship

The algebraic relationship between ω and k, expressed as $\omega = f(k)$, is nemed the wave *dispersion relationship*.

F.4.1 Non-dispersive waves

If ω has a linear dependence on k, implying that $\omega \propto k$, the wave is said to be *non-dispersive*. In this case the waves propagate with the same phase velocity, see (589) and (594), irrespective of the wave's wavelength.

F.4.2 Dispersive waves

If ω does not depend linearly on k, waves with different wavelengths will propagate with different phase speeds. In this case, the wave is said to the *dispersive*.

F.5 Group speed

For dispersive waves, energy does not propagate with a single trigonometric wave, but with the "total" contribution from all waves forming the wave field. The "total contribution" is given by the group speed, representing the total contribution from all wave components forming a wave field. Thus, the group speed is in many respects a more central quantity than the properties of individual wave components.

The group speed is given by the expression

$$c_g = \frac{\partial \omega}{\partial k} \tag{601}$$

Nice illustrations of the relationship between the phase speed and group speed are given here: `http://www.isvr.soton.ac.uk/spcg/tutorial/tutorial/Tutorial_files/Web-further-dispersive.htm+`.

F.5.1 Derivation

The expression for the group speed, (601), can be understood by considering two waves with nearly equal wave numbers and wave frequencies, so $k_1 \approx k_2$ and $\omega_1 \approx \omega_2$ in the following expression:

$$\zeta = \overline{\zeta}_0 \cos(k_1 x - \omega_1 t) + \overline{\zeta}_0 \cos(k_2 x - \omega_2 t) \tag{602}$$

From the identity

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b \tag{603}$$

it follows that

$$\cos(a + b) + \cos(a - b) = 2 \cos a \cos b \tag{604}$$

With

$$a = \frac{1}{2}(k_1 + k_2)x - \frac{1}{2}(\omega_1 + \omega_2)t \tag{605}$$

and

$$b = \frac{1}{2}(k_2 - k_1)x - \frac{1}{2}(\omega_2 - \omega_1)t \tag{606}$$

expression (602) be put in the form

$$\zeta = 2\overline{\zeta}_0 \cos\left[\frac{1}{2}(k_1 + k_2)x - \frac{1}{2}(\omega_1 + \omega_2)t\right] \cos\left[\frac{1}{2}(k_2 - k_1)x - \frac{1}{2}(\omega_2 - \omega_1)t\right] \tag{607}$$

Since the waves are assumed to have similar (albeit not identical) wave numbers and frequencies, it follows that $k_1 \approx k_2$ and $\omega_1 \approx \omega_2$. We can now define

$$k = \frac{k_1 + k_2}{2}, \qquad \omega = \frac{\omega_1 + \omega_2}{2}, \qquad \Delta k = k_2 - k_1, \qquad \Delta\omega = \omega_2 - \omega_1 \tag{608}$$

Upon insertion into (607), one obtains

$$\zeta = 2\,\overline{\zeta}_0 \cos\left(\frac{1}{2}\Delta k\, x - \frac{1}{2}\Delta\omega\, t\right) \cos(kx - \omega t) \tag{609}$$

A variant of experssion (609) is illustrated in Fig. 17, showing the modulation of the combined $\mathbf{M_2}$ and $\mathbf{S_2}$ tidal constituents in Bergen, Norway. The basic periods for the two tidal constituents are about around 12 hr, yielding a slowly varying (beating) period of about 14 days. In this example, the group speed is represented by the fortnightly signal, *not* the speed of the individual wave components.

Expression (609) is a wave that propagates as a standard wave on the form $\cos(kx - \omega t)$ with the phase speed $c = \omega/k$, but where the amplitude $2\,\overline{\zeta}_0$ is modulated with a slowly varying wave given by $\cos(\Delta k\, x/2 - \Delta\omega\, t/2)$ with (a long) wavelength of $4\pi/\Delta k$ and with a (long) period of $4\pi/\Delta\omega$.

The modulation wave propagate with a speed given by λ/T, see experssion (589). The latter ratio can be expressed in terms of wave number k and frequency ω:

$$\frac{\Delta\omega}{\Delta k} \tag{610}$$

or, for small $\Delta\omega$ and Δk,

$$\frac{\partial\omega}{\partial k} \tag{611}$$

It is the latter quantity that is the wave's group speed, which can be viewed as the "total contribution" from all wave components. Propagation of the wave's energy is thus associated with the collective speed of the individual wave components, not the speed of any individual wave component.

G Resources

Particularly useful resources are shown in red.

Equinoxes, Solstices, Perihelion, and Aphelion:
`http://aa.usno.navy.mil/data/docs/EarthSeasons.php`

Global Extreme Sea Level Analysi
`https://www.gesla.org`

Kowalik & Luick, *The Oceanography of Tides*, Fairbanks, January, 2013:
`http://www.uaf.edu/files/sfos/Kowalik/tide_book.pdf`

Lunar Perigee and Apogee Calculator:
`http://www.fourmilab.ch/earthview/pacalc.html`

Norwegian Mapping Authority, Sea Level and Tides:
`https://www.kartverket.no/sehavniva`

Norwegian Mapping Authority, Tide Tables for the Norwegian Coast and Svalbard:
`https://www.kartverket.no/til-sjos/se-havniva/lar-om-tidevann-og-vannstand/tabellar-for-tidvatn/`

NOVAS, Positional Astronomy:
`http://aa.usno.navy.mil/software/novas/novas_info.php`

Permanent Service for Mean Sea Level:
`http://www.psmsl.org/data`

Pugh & Woodworth, *Sea-Level Science*, Cambridge, 2014:
`https://doi.org/10.1017/CBO9781139235778`

Tides and Currents Glossary:
`https://tidesandcurrents.noaa.gov/glossary.html`

TSOFT, A software package for the analysis of Time Series and Earth Tides:
`http://seismologie.be/en/downloads/tsoft`

H Formulas and identities

H.1 Binomial theorem for rational exponents

For all rational[20] numbers r, the following identity exists

$$(1+x)^r = 1 + r\,x + r\frac{r-1}{2}x^2 + r\frac{r-1}{2}\frac{r-2}{3}x^3 + ... \tag{612}$$

For a derivation, see e.g.
`http://www.trans4mind.com/personal_development/mathematics/series/binomialProofAllAlgebra.htm`.

For $|x| \ll 1$, the linear and quadric approximation of (612) becomes

$$(1+x)^r = 1 + r\,x \tag{613}$$

and

$$(1+x)^r = 1 + r\,x + \frac{r(r-1)}{2}x^2 \tag{614}$$

H.2 Some trigonometric identities

Negative angle identities

$$\sin(-x) = -\sin x \tag{615}$$

$$\cos(-x) = \cos x \tag{616}$$

$$\tan(-x) = -\tan x \tag{617}$$

Pythagorean identity

$$\sin^2 a + \cos^2 a = 1 \tag{618}$$

Sum and difference identities

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b \tag{619}$$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b \tag{620}$$

Product identities

$$2\,\sin a \sin b = \cos(a-b) - \cos(a+b) \tag{621}$$

$$2\,\cos a \cos b = \cos(a-b) + \cos(a+b) \tag{622}$$

$$\sin^2\,a = \frac{1}{2}(1 - \cos\,2a) \tag{623}$$

$$\cos^2\,a = \frac{1}{2}(1 + \cos\,2a) \tag{624}$$

[20] A rational number is any number that can be expressed as the quotient or fraction p/q of two integers, with the denominator q not equal to zero.

Product sum identities

$$\sin a \pm \sin b = 2\,\sin\left(\frac{a \pm b}{2}\right)\cos\left(\frac{a \mp b}{2}\right) \tag{625}$$

$$\cos a + \cos b = 2\cos\left(\frac{a + b}{2}\right)\cos\left(\frac{a - b}{2}\right) \tag{626}$$

$$\cos a - \cos b = -2\sin\left(\frac{a + b}{2}\right)\sin\left(\frac{a - b}{2}\right) \tag{627}$$

Double angle identities

$$\sin 2a = 2\,\sin a \cos a \tag{628}$$

$$\cos 2a = \cos^2 a - \sin^2 a \tag{629}$$

$$\cos 2a = 2\cos^2 a - 1 \tag{630}$$

$$= 1 - 2\sin^2 a \tag{631}$$

Triple angle identity

$$\sin^3 a = \frac{1}{4}(3\sin a - \sin 3a) \tag{632}$$

Half-angle identities

$$\cos a = \frac{1 - \tan^2(a/2)}{1 + \tan^2(a/2)} \tag{633}$$

Integrals

$$\int \sin 2x\,dx = -\frac{1}{2}\,\cos 2x + C \tag{634}$$

$$\int \cos 2x \sin x\,dx = \frac{\cos x}{2} - \frac{\cos 3x}{6} + C \tag{635}$$

$$\int \cos^2 x\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + C \tag{636}$$

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C \tag{637}$$

$$\int \sin x \cos x\,dx = -\frac{\cos 2x}{4} + C \tag{638}$$

Lowest order approximations

$$\sin x \approx x \quad \text{for} \quad x \ll 1 \tag{639}$$

$$\cos x \approx 1 \quad \text{for} \quad x \ll 1 \tag{640}$$

Law of cosines

For any triangle with sides a, b and c, with the angle between a and b being C, the following relationship holds:

$$c^2 = a^2 + b^2 - 2\,ab\cos C \tag{641}$$

Cotangent formula

On a spherical triangle, let the four elements

side – angle – side – angle

lie adjacent to each other. If the elements are referred to as

outer side (OS) – inner angle (IA) – inner side (IS) – outer angle (OA)

then the formula

$$\cos(\mathrm{IS})\cos(\mathrm{IA}) = \sin(\mathrm{IS})\cot(\mathrm{OS}) - \sin(\mathrm{IA})\cot(\mathrm{OA}) \tag{642}$$

is valid. An example of the location of the four spherical triangle elements is illustrated in Fig. 37.

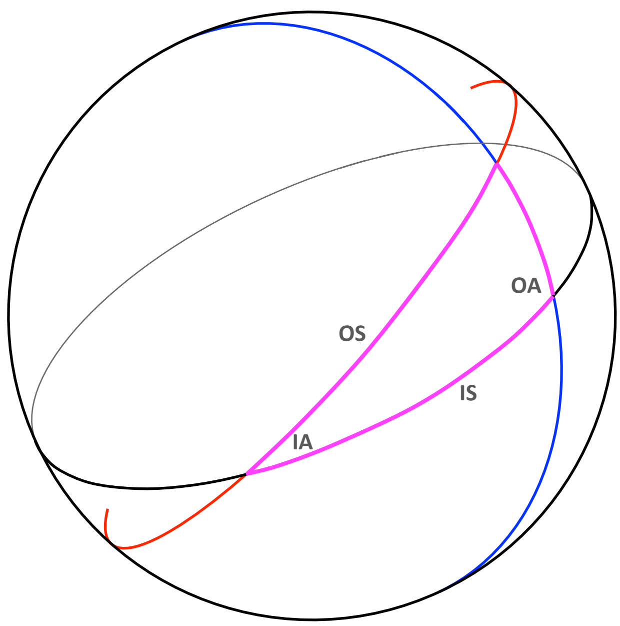

Figure 37: Illustration of one out of three ways to locate the four adjacent elements 'outer side' (OS) – 'inner angle' (IA) – 'inner side' (IS) – 'outer angle' (OA) on a spherical triangle (outlined in magenta). This ordering is consistent with the cotangent formula written in the form (642).

Derivative of $\arctan x$

$$\frac{d\,\arctan x}{d\,x} = \frac{1}{1+x^2} \tag{643}$$

I Norwegian dictionary

The following dictionary is an excerpt (with a few additions) from the Norwegian Mapping Authority's online dictionary,
`https://www.kartverket.no/Systemsider/Ordbok`.

Altimeter Høydemåler, instrument til å måle høyden over et bestemt nivå.

Amfidromisk punkt Et punkt uten tidevann. Et slikt punkt finnes sørvest for Egersund.

Apogeum Det punktet i månens eller en jordsatellitts bane som ligger lengst borte fra jorda. Motsatt av perigeum.

Buegrad Enhet ved måling av sirkelbue. Ved en buegrad er vinkelen mellom vinkelbena i sirkelbuen 1 grad. En sirkel deles inn i 360 grader (360°), så en buegrad er 1/360 av en sirkels omkrets.

Brakkvann Sjøvann med saltholdighet mellom ca. 0,5 og 17,0 promille.

Corioliskraften En såkalt fiktiv kraft som påvirker et legeme som beveger seg i forhold til jordoverflaten. Corioliskraften skyldes jordas rotasjon. En som står på jordoverflaten vil oppleve at et legeme får en avbøyning mot høyre for bevegelsesretningen på den nordlige halvkule og mot venstre på den sørlige halvkule.

Deklinasjon Astronomisk koordinat som sammen med rektascensjon gir entydig posisjon. Vinkelen ved observasjonsstedet (himmelkulens sentrum) mellom en linje som går gjennom himmellegemet og himmelkulens ekvatorplan. Deklinasjonen måles som den delen av storsirkelen som ligger mellom himmellegemet og ekvatorplanet. Deklinasjonen er positiv når himmellegemet er nord for ekvator og negativ når det er sør for ekvator.

Dobbelt høyvann Høyvann som består av to maksima av tilnærmet samme høyde og atskilt av en kortere periode med noe lavere vannstand.

Dobbelt lavvann Lavvann som består av to minima av tilnærmet samme høyde og atskilt av en kortere periode med noe høyere vannstand.

Drivlegeme Flytende objekt (for eksempel en flaske, stang, bøye, strømkors eller drivkort) som er beregnet på å bestemme strømmens retning og fart ved gjentatt bestemmelse av legemets plass mens det flyter fritt.

Dybde Loddrett avstand fra et gitt referansenivå ned til bunnen eller til et annet objekt.

Ebbe Synkende tidevann.

Ekliptikk Jordbaneplanets skjæringssirkel med himmelkulen, eller solbanen sett fra jorden.

Ekvidistanse Den loddrette avstanden mellom to nabohøydekurver.

Ellipsoide Ellipsoide er ein geometrisk form. Jordkloden har tilnærma form som ein ellipsoide, og ein nyttar ein ellipsoide-modell til å rekne ut ei matematisk høgde over havet.

Ellipsoidisk høyde Et punkts avstand fra referanseellipsoiden, målt langs ellipsoidenormalen.

Fjordoverføring Måling av høydeforskjell over fjord, vann eller elv, der vanlige nivelleringsmetoder blir erstattet av synkrone avlesninger med spesielt nøyaktig instrumentering og observasjonsprosedyre.

Flø Stigende tidevann.

Frihøyde Høyde for fri passasje. Frihøyde er mer omfattende enn friseilingshøyde. Frihøyde omfatter også minste høyde i tunneler, underganger, vei-/jernbanebruer, foruten minste høyde av kraftlinjer over terrenget, bygning osv.

Friseilingshøyde Minste seilingshøyde under bru, luftspenn og lignende fra et gitt referansenivå. Fra 1. januar 2000 er høyeste astronomiske tidevann (HAT) referansenivå for friseilingshøyde.

Geodesi Geodesi er vitenskapen om jordens form, bevegelse, tyngdefelt og endringer i disse størrelsene.

Geodetisk datum Parametre som definerer et koordinatsystems plassering og orientering i forhold til jorda. Det skilles gjerne mellom et horisontalt datum, for angivelse av nord og øst-koordinater, og et vertikalt datum for angivelse av høyder. I dag brukes også begrepet referansesystem i stedet for datum.

Geoide Geoiden er en referansehøyde som representerer den teoretiske høyden på havet i fravær av tidevannsbevegelse, havstrømmer eller bølger. Den følger havets tenkte forlengelse under kontinentene. For å kunne danne denne referansehøyden må man kjenne jordens tyngdefelt, som ikke er likt over alt. Geoiden er et helt nødvendig utgangspunkt for alle nøyaktige høydemålinger, som de svært viktige klimavariablene havnivå, havsirkulasjon og istykkelse. Aktiviteter som krever nøyaktige høydemålinger, som konstruksjon og byutvikling, er avhengig av geoiden som utgangspunkt.

Geoidemodell En modell som angir hvor høyt geoiden ligger i forhold til en valgt ellipsoide.

Geopotensialtallet Differanse i tyngdepotensial mellom geoiden og et punkt over geoiden.

Geosentrisk Med jorda som midtpunkt, i motsetning til heliosentrisk, som har sola som midtpunkt.

Gjentaksintervall Beregninger av hvor hyppig en stormflo av en viss størrelse statistisk sett vil opptre. Høyvann (lavvann) med x års gjentaksintervall: Vannstanden forventes i gjennomsnitt å bli så høy (lav) en gang i løpet av x antall år. Kalles også returverdi eller returnivå.

GLOSS Global Sea Level Observing System (GLOSS) er et internasjonalt program opprettet under beskyttelse av Joint Technical Commission for Oceanography and Marine Meteorology (JCOMM) (www.jcomm.info/) tilhørende World Meteorological Organisation (WMO) og Intergovernmental Oceanographic Commission (IOC). GLOSS skal være en bidragsyter i å etablere globale og regionale målenett for vannstand av høy kvalitet, der dataene brukes til forskning innen klima, vannstand og oseanografi. Den viktigste delen til GLOSS består av et globalt målenett (hovednettverk) der 290 vannstandsmålere inngår. Mer informasjon på www.gloss-sealevel.org.

GMT Greenwich (middel-) tid, det samme som UT (Universal Time).

Greenwich Bydel i London med observatorium grunnlagt 1675. Lengdesirkelen gjennom observatoriet har siden 1883 vært ansett som nullsirkelen (nullmeridianen). Observatoriet ble ca. 1950 flyttet til Herstmonceaux i Sussex, men Greenwich-meridianen gjelder fremdeles som nullmeridianen. Greenwich Mean Time, forkortet GMT, middelsoltid i Greenwich, regnes etter 1925 fra midnatt (tidligere regnet fra middag).

Grunnlinjepunkt Koordinatbestemt punkt på de ytterste nes og skjær som stikker opp av havet ved lavvann.

Harmonisk analyse Matematisk metode som løser opp en (observert) kurve i en rekke cosinuskurver. Ofte brukt til analyse av vannstandskurver. Resultatet kan settes sammen etter et visst mønster for å beregne tidevannet i et vilkårlig tidspunkt.

Harmonisk konstant Amplitude og faseforskyvning for en harmonisk konstituent beregnet for et bestemt sted. Beregnes ved hjelp av harmonisk analyse.

Harmonisk konstituent En av flere cosinusfunksjoner i den matematiske modellen for tidevannet. Amplitude og fase for konstituenten beregnes ved hjelp av harmonisk analyse. Frekvensen er beregnet fra teorien for likevektstidevann.

HAT Høyeste astronomiske tidevann (HAT) er høyeste mulige vannstand uten værets virkning. Det vil si uten påvirkning fra blant annet vind, lufttrykk og temperatur.

Havnetid Tidsrommet fra månen passerer stedets meridian (eller annen referansemeridian) til første påfølgende høyvann inntreffer. Havnetiden har et karakteristisk forløp gjennom en spring-nipp-periode. Vanligvis angis midlere havnetid.

Havnivå Havets gjennomsnittsnivå målt over en lang periode, slik at variasjoner forårsaket av tidevannskrefter og vær ikke påvirker resultatet. Havnivået er den trege komponenten av vannstand.

Havstrøm Havstrømmer påvirker havnivået. Jo raskere overflatestrømmene i havet går, jo mer vil de bule opp og danne en ujevn havoverflate, og havnivået endres.

Horisontalt datum Et referansesystem som angir nord- og øst-koordinater, i motsetning til et vertikalt datum (høydedatum), som angir høyder.

Href Høydereferansemodell som gir forskjellen mellom høyder i NN1954 og EUREF89. En høydereferansemodell angir høydedifferansen mellom en ellipsoide og en geoide.

Hydrografisk nivellement Overføring av middelvann (og andre nivå) fra et sted med en lang observasjonsrekke (av vannstand) til et sted med kort observasjonsrekke. Metoden benytter samtidige vannstandsobservasjoner på de to stedene. Brukes også ved sekundærhavn-beregning.

Høydedatum Et referansesystem som angir høyder (også kalt vertikalt datum), i motsetning til et horisontalt datum, som angir nord- og øst-koordinater.

Høydereferansemodell Modell som angir høydedifferansen mellom en ellipsoide og en geoide. Href er høydereferansemodell som gir forskjellen mellom høyder i NN1954 og EUREF89.

Høydesystem Vertikalt eller geodetisk datum, og et nett av utvalgte fastmerker som er høydebestemt i dette datumet.

Høyvann Høyeste vannstand på et sted i løpet av én tidevannsperiode, også kalt flo.

Isoraki Linje på et tidevannskart som går gjennom steder der høyvann inntreffer på samme tid. Tidspunktene for høyvann er angitt i forhold til månens passasje gjennom én bestemt meridian, for eksempel 0° E. Isorakiene gir oversikt over tidevannsbølgens forplantning.

Jevndøgn Dato når solen under sin årlige bevegelse langs ekliptikken kommer til skjæringspunktet (jevndøgnspunktene) mellom ekliptikken og himmelens ekvator. Solens deklinasjon er da null, det vil si at dag og natt er like lange. Vårjevndøgn inntreffer 20. eller 21. mars og høstjevndøgn 22. eller 23. september.

K1 Et ledd i en matematisk rekke som beskriver tidevannsvariasjonene. Hvert ledd i rekken representerer periodiske virkninger på tidevannet fra måne og sol, og er gitt med en amplitude og fase. K1 er en lunisolar (tilknytning både til månen og solen), heldaglig konstituent. Sammen med konstituenten O1 uttrykker den effekten av månens deklinasjon. Mens den sammen med konstituenten P1 uttrykker effekten av solens deklinasjon.

K2 Denne konstituenten modulerer amplituden og frekvensen til M2 og S2 for deklinasjonseffekten for henholdsvis månen og solen.

L2 Månen går i en elliptisk bane rundt jorden, der månens banehastighet varierer i ellipsebanen. Konstituenten L2 modulerer sammen med konstituenten N2 amplituden og frekvensen til M2 for effekten av denne variasjonen.

Landheving Landheving innebærer at landjorden hever seg. Landhevingen i Skandinavia skyldes i hovedsak at isen som trykket landjorden ned under siste istid, har smeltet. Dermed hever landet seg. Landhevingen varierer fra sted til sted, men er på det meste nesten 7 millimeter i året i forhold til jordens sentrum. I forhold til havnivå er landhevingen ca 2 millimeter lavere siden havnivået også stiger.

LAT Laveste astronomiske tidevann (LAT) er laveste mulige vannstand uten værets virkning. Det vil si uten påvirkning fra blant annet vind, lufttrykk og temperatur.

Lavvann Laveste vannstand på et sted i løpet av én tidevannsperiode, også kalt fjære.

Likevektstidevann Modell av tidevannet som forutsetter at jordkloden er fullstendig dekket av et jevntykt vannlag og der vannet antas å være uten friksjon og treghet. Modellen gir ikke korrekte verdier for tidevannsvariasjonene, men inngår som en viktig del av det teoretiske grunnlag for forståelsen av tidevann.

M2 Et ledd i en matematisk rekke som beskriver tidevannsvariasjonene. Hvert ledd i rekken representerer periodiske virkninger på tidevannet fra måne og sol, og er gitt med en amplitude og fase. M2 er det dominerende tidevannsbidraget fra månen.

Mellomeuropeisk tid Tidssone som ligger en time foran Greenwich middeltid, dvs. GMT + 1 time. Fellestid for de fleste europeiske land.

Meridian Meridian eller lengdegrad er en tenkt linje i jordens koordinatsystem som går mellom Nordpolen og Sydpolen. Meridianene står vinkelrett på ekvator. Nullmeridianen går gjennom Greenwich i London.

Middel høyvann Gjennomsnitt av alle høyvann i en 19-årsperiode.

Middel lavvann Gjennomsnitt av alle lavvann i en 19-årsperiode.

Middel nipp høyvann Gjennomsnitt av alle høyvannene i nipperioder i løpet av 19 år.

Middel nipp lavvann Gjennomsnitt av alle lavvannene i nipperioder i løpet av 19 år.

Middel spring høyvann Gjennomsnitt av alle høyvannene i springperioder i løpet av 19 år.

Middel spring lavvann Gjennomsnitt av alle lavvannene i springperioder i løpet av 19 år.

Middelvann Middelvann (MV) er gjennomsnitt av alle vannstandsmålinger i en 19-årsperiode.

Målernull Nullpunkt for en vannstandsmåler. Er i prinsippet et tilfeldig valgt nivå.

N2 Sammen med konstituenten L2, modulerer N2 amplituden og frekvensen til M2 for effekten av variasjonen i månens banehastighet på grunn av månens elliptiske bane rundt jorden.

Nasjonalt høydesystem Nasjonalt vedtatt høydesystem. NN1954 er eksempel på et nasjonalt høydesystem.

Nipp Nipp får man når tidevannet er på sitt laveste. Det skjer når tidevannskreftene fra månen og sola virker mest mulig sammen.

Nivellement Målemetode for nøyaktig høydebestemmelse, relatert til loddlinjen. Målingen skjer ved å foreta horisontale sikt med et nivellerinstrument mot loddrett oppstilte nivellerstenger eller meterskalaer. Ved å summere høydeforskjellene mellom mange horisontale sikteliner får vi nøyaktige høyder langt inne i landet.

NN1954 Normal Null 1954 (NN1954) er navn på det nasjonale høydesystemet fra 1954 som fortsatt er i bruk i Norge. NN1954 er også fysisk knyttet til et bestemt fastmerke ved Tregde vannstandsmåler (nær Mandal). Høyden på dette fastmerket er basert på en utjevning fra 1954 av middelvannstandsberegningene for vannstandsmålerne i Oslo, Nevlunghavn, Tregde, Stavanger, Bergen, Kjølsdal og Heimsjø. NN1954 avløses innen år 2015 av Normal Null 2000 (NN2000).

NN2000 NN2000 er Norges nye høydesystem som innføres gradvis fram til 2016/2017. Høydesystemet er den referansen som ligger til grunn når man angir hvor mange meter over havet (moh.) for eksempel et fjell eller en innsjø ligger.

NNN1957 Nord-norsk null 1957 (NNN1957) var vertikalt datum for det nasjonale høydesystem i Nord-Norge, nord for Tysfjord og i Lofoten, fram til 1996. NNN1957 ble brukt som navn på både det vertikale datumet og på høydesystemet. I 1957 ble det innført et nytt utgangsnivå for høyder i Nord-Norge. På den tiden var det en rekke veibrudd med ferjeforbindelser mellom Fauske og Narvik. Følgelig var det ikke noen direkte forbindelse mellom nivellementslinjene nord for Narvik og det sørnorske nivellementsnettet. Det var derfor behov for å etablere et fundamentalpunkt for Nord-Norge. Det er upraktisk å operere med to offisielle høydesystemer. For alle tekniske formål må vi kunne si at NN1954 og NNN1957 faller sammen. Det ble derfor vedtatt at betegnelsen NNN1957 skulle falle bort fra 1. januar 1996.

Normaltid Et lands offisielle tid, i Norge brukes mellomeuropeisk tid.

O1 Et ledd i en matematisk rekke som beskriver tidevannsvariasjonene. O1 er en lunar (tilknytning til månen) heldaglig konstituent. Sammen med konstituenten K1 uttrykker den effekten av månens deklinasjon.

P1 Et ledd i en matematisk rekke som beskriver tidevannsvariasjonene. Hvert ledd i rekken representerer periodiske virkninger på tidevannet fra måne og sol, og er gitt med en amplitude og fase. P1 er solar (tilknytning til solen) heldaglig konstituent. Sammen med konstituenten K1 uttrykker den effekten av solens deklinasjon.

Predikert tidevann Ved hjelp av harmonisk analyse bestemmes konstantene i en modell for tidevannet. Med modellen kan tidevannet beregnes for et vilkårlig, fremtidig tidspunkt (predikeres). Tidevannstabeller er basert på predikert tidevann.

Perigeum Punkt i månens eller en jordsatellitts bane som ligger nærmest jorden. Perigeum er motsatt av apogeum.

Potensialflate Flate hvor potensialet har samme verdi. I litteratur også kalt ekvipotensialflate.

Presisjonsnivellement Nivellement som setter strenge krav til utførelse og kontroll. Det er vanlig å sette krav om at en strekning skal måles både fram og tilbake, og at avviket skal være under en viss toleranse.

Referanseflate Entydig definert flate som målinger og beregninger henføres til. Ellipsoide, geoide, kvasigeoide og sjøkartnull er eksempler på referanseflater for høyde/dybde.

Referansenivå Høyder og dybder må refereres til et bestemt nullnivå.

Returnivå Beregnet nivå vannstanden sjelden overstiger. Returnivå med 5 års gjentaksintervall: Statistisk sett forventer man så høy vannstand en gang i løpet av 5 år. For et gitt år er det 1/5 = 20

ROS-analyse Risiko- og sårbarhetsanalyse. Risiko = usikkerhet knyttet til forekomst og alvorlighet av uønskede hendelser. Sårbarhet = et uttrykk for et systems manglende evne til å fungere og oppnå sine mål når det utsettes for påkjenninger. Les mer

S2 Et ledd i en matematisk rekke som beskriver tidevannsvariasjonene. Hvert ledd i rekken representerer periodiske virkninger på tidevannet fra måne og sol, og er gitt med en amplitude og fase. S2 er det dominerende tidevannsbidraget fra solen.

Satellittaltimetri Målinger av havnivå som utføres ved at en satellitt måler sin egen høyde over jordkloden ved hjelp av radar eller laser. Dersom satellittens høyde er kjent i en referanseramme, er det mulig å bestemme høyden på havoverflaten i den samme referanserammen ved å trekke den målte avstanden fra satellitthøyden.

Sekundærhavn Havn der tidevannstabeller vanligvis angir tidsdifferansen (av og til også høydedifferansen) for høyvann og lavvann i forhold til en bestemt standardhavn.

Sjøkartnull Nullnivå for dybder i sjøkart og høyder i tidevannstabeller. Sjøkartnull er fra 1. januar 2000 lagt til laveste astronomiske tidevann (LAT). Langs Sørlandskysten og i Oslofjorden er tidevannsvariasjonene små i forhold til værets virkning på vannstanden (vind, lufttrykk og temperatur). Sjøkartnull er derfor av sikkerhetsmessige grunner lagt 20 cm lavere enn LAT langs kysten fra svenskegrensen til Utsira og 30 cm lavere enn LAT i indre Oslofjord (innenfor Drøbaksundet).

Spring Spring får man når tidevannet er på sitt høyeste. Det skjer når tidevannskreftene fra månen og sola virker mest mulig sammen. Spring inntrer i de fleste steder i Norge ved hver nymåne og fullmåne, omtrent hver fjortende dag.

Standardhavn Sted der harmoniske konstanter er bestemt ved registrering av vannstand over et lengre tidsrom. I tidevannstabellen oppgis alle tidspunkt og høyder for høyvann og lavvann for standardhavnene.

Stangnull Nullpunkt for en tidevannsstang.

Stormflo Når værets virkning på vannstanden er spesielt stor, kalles det stormflo. Dette skyldes som regel lavt lufttrykk og kraftig vind som presser vannet inn mot kysten. Dersom en stormflo faller sammen med en springperiode, kan man få ekstra høy vannstand.

Strømkart Kart over sjøområdene som viser horisontal strømhastighet. Ofte angis strømforholdene i forhold til tidspunktet for høyvann i en standardhavn. Strømmen presenteres ved piler der fart og retning symboliseres.

Strømmåler Instrument til måling av vannmassers fart og/eller retning.

Tidejord Periodisk deformasjon av den delvis elastiske jordkroppen forårsaket av himmellegemenes (i første rekke solens og månens) gravitasjonstiltrekning på den roterende jorden. Tilsvarende deformasjon i havet kalles tidevann.

Tidevann Vannstandsendringer som skyldes variasjoner i tiltrekningskreftene fra månen og sola på jorda.

Tidevannets alder Tidsrommet fra ny- eller fullmåne til det neste spring høyvann. Angis ofte oftest ved hjelp av harmoniske konstanter.

Tidevannsbølge Lang bølge forårsaket av tidevannskraften fra sol og måne. Bølgelengden kan bli flere tusen kilometer. Den dannes ute på de store havområder og forplanter seg mot kontinentene der den forårsaker høyvann og lavvann. Bølgen kan gjennomgå betydelige endringer som følge av hindringer (landmasser og bunnformasjoner) på vei mot kysten.

Tidevannskart Kart som for én harmonisk konstituent viser tidevannets karakter over større eller mindre områder. Vises ved linjer som går gjennom steder der høyvann inntreffer samtidig (isorakier) eller gjennom steder med samme tidevannsamplitude.

Tidevannsklokke Et urverk som følger månen og viser når det er høyvann eller lavvann.

Tidevannsperiode Tid mellom to påfølgende høyvann eller lavvann. I Norge er tidevannsperioden i middel 12 timer og 25 minutter.

Tidevannsstang Gradert stang (vanligvis centimeterinndeling) som anbringes vertikalt i sjøen for avlesning av vannstanden. Skalaen på en tidevannsstang er vanligvis innmålt i forhold til et vannstandsmerke.

Tidevannsstrøm Strøm i havet forårsaket av tidevannskreftene fra sol og måne. På åpent hav vil tidevannsstrømmen rotere 360° i løpet av en tidevannsperiode. Nær kysten og i fjorder og sund vil tidevannsstrømmen ha en nesten rett bevegelse frem og tilbake.

Tidevannsutbuling Se *utbuling*.

Topografi Topografi er en beskrivelse av jordoverflatens terreng og synlige objekter, slik som høyde, vegetasjon, hav, innsjøer, bebyggelse og veier.

Tyngdekraft Tiltrekningskraft som virker mellom alle partikler med masse i universet, også kalt gravitasjonskraft.

Tyngdepotensial Summen av gravitasjons- og sentrifugalpotensial.

Tørrfall Sjøkart: Del av kysten som ligger i dybdeområdet fra middel høyvann til 0.5 m under sjøkartnull. Landkart: Sandbanker og avleiringer i elver. Oversvømmes ved høy vannføring.

Utbuling Opphopning av vann på jorden grunnet månens (og solens) tiltrekning. Hadde ikke jorden rotert (eller dersom jordens rotasjon var tilstrekkelig lav), ville utbulingen ligge på sentrallinjen mellom jord og måne. Grunnet jordens raske rotasjon, i kombinasjon med friksjon mellom jord og hav, ligger utbulingen til venstre for senterlinjen – eller at den ligger foran månen – når jord-månesystemet betraktes fra nord.

Vannstand Høyden av vannflaten på et bestemt sted på et gitt tidspunkt. For havet påvirkes vannstanden av tidevann og værets virkning (vind, lufttrykk, med mer).

Vannstandsmerke Fastmerke i terrenget brukt som referansemerke for vannstand. Gjerne knyttet til et nivellementsnett.

Vertikalt datum Referansesystem som angir høyder (også kalt høydedatum), i motsetning til et horisontalt datum, som angir nord- og øst-koordinater.

Vref Betegnelse på en høydereferansemodell som angir høydeforskjellen mellom NN1954 og EUREF89, beregnet fra en geoidemodell.

Værets virkning Påvirkning på vannstanden som skyldes vind, lufttrykk og temperatur. Ekstra store værbidrag kalles stormflo.

J Acknowledgement

The author would like to acknowledge comments and suggestions from the following students and readers: Yannick Kern (student, fall 2016), Aleksander Dürr Libæk and Øystein Markhus Breiteig (student, fall 2018), and Victor Cesar Martins de Aguiar (student, fall 2019).

K Index